Generalizations of Kloosterman Sums

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December 15, 2006

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- Problem: find a good bound for the sum

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Applications:

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- Coding theory and graph theory

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Weil's bound:

$$\left|\sum_{x \in k^\star} \psi(ax + \frac{b}{x})\right| \leq 2\sqrt{q}$$

Higher dimensional Kloosterman sums

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Deligne:

$$|\mathrm{Kl}_{n,\psi}(t)| \leq nq^{(n-1)/2}$$

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Katz:

$$|\mathrm{Kl}_{n,\psi,\chi_1,\ldots,\chi_n,a_1,\ldots,a_n}(t)| \leq (a_1 + \cdots + a_n)q^{(n-1)/2}$$

Exponential sums

$$f \in k[x_1, \ldots, x_n] \text{ of degree } d$$

$$S_\psi(f) = \sum_{x_1, \ldots, x_n \in k} \psi(f(x))$$

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$$\psi_m(t) = \psi(\mathrm{Trace}_{k_m/k}(t))$$

The L-function

$$L(\psi, f, T) = \exp \sum_{m \geq 1} \frac{S_{\psi,m}(f)}{m} T^m \in 1 + T\mathbb{Q}(\zeta_p)[[T]]$$

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$$L(\psi, f, T) = \prod_{i=n}^{2n} P_i(T)^{(-1)^{i+1}}$$

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Deligne:

$$P_i(T) = \prod_{j=1}^{d_i} (1 - \alpha_{i,j} T)$$

where $\alpha_{i,j}$ is an algebraic integer, pure of integer weight $w_{i,j} \leq i$:

$$|\alpha_{i,j}| \leq q^{w_{i,j}/2}$$

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Need to estimate:

- the weights $w_{i,j}$
- the degrees d_i

Known results

Deligne (Weil I): If $\gcd(d, p) = 1$ and the highest degree form of f is non-singular, then $d_i = 0$ for $i > n$, $d_n = (d-1)^n$ and $w_{n,j} = n$, so

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Katz: If $\gcd(d,p)=1$ and the highest degree form of f has singular locus of dimension $\epsilon \geq 0$, then $d_i=0$ for $i>n+1+\epsilon$, so

$$|S_\psi(f)|\leq Dq^{(n+\epsilon+1)/2}$$

Main result

Let $f = f_d + f_{d'} + h$ with f_d homogeneous of degree d, $f_{d'}$ homogeneous of degree $d' < d$ and h of degree $< d'$, and $f_d = g_1^{a_1} \cdots g_r^{a_r}$, $\deg(g_i) = e_i$ such that $f_d f_{d'}$ defines a divisor with normal crossings in $\mathbb{P}^{n-1}$ and $\gcd(p, dd' a_1 \cdots a_r) = 1$. Then

- $d_i = 0$ for $i > n$
- $d_n = C(d, d', r, e_1, \ldots, e_r)$
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In particular,

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ie. a rank one smooth ℓ-adic sheaf $\mathcal{L}_f$ on $\mathbb{A}^n$ Every k-rational point $x \in k^n$ gives

$$\pi_1(\mathrm{Spec}(k(x))) \to \pi_1(\mathbb{A}^n)$$

so we get an element $F_x \in \pi_1(\mathbb{A}^n)$, well defined up to conjugation, on which the character $\mathcal{L}_f$ takes the value $\psi(f(x))$.

On the other hand, we have cohomology groups

$$H_c^i(\mathbb{A}^n, \mathcal{L}_f)$$

vanishing for $i < n$ and $i > 2n$, on which $\mathrm{Gal}(\bar{k}/k)$ acts.

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This formally implies

$$L(\psi, f, T) = \prod_{i=n}^{2n} (1 - T \det(F | H^i_c(\mathbb{A}^n, \mathcal{L}_f)))^{(-1)^{i+1}}$$

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$$\sum_{x_1,\ldots,x_n \in k} \psi(x_1^d + \cdots + x_n^d) = \Big(\sum_{x_1 \in k} \psi(x_1^d)\Big) \cdots \Big(\sum_{x_n \in k} \psi(x_n^d)\Big)$$

so the result follows from Weil's

Families of exponential sums

In our case:
$\mathcal{P}_i =$ affine space parameterizing all homogeneous polynomials of degree e_i
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$U \subset \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \times \mathcal{P}$ open subset parameterizing all $(r+1)$-tuples $(g_1, \ldots, g_r, h)$ such that $g_1^{e_1} \cdots g_r^{e_r} + h$ is "good"

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The fact that $g_1 \cdots g_e f_{d'}$ defines a divisor with normal crossings implies that the $H_c^i(\mathbb{A}^n, \mathcal{L}_f)$ vary smoothly on U, in particular the d_i's and the $w'_{i,j}s$ are constant.

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and in fact it is the restriction to a linear subspace of U of the complex parameterizing the cohomology of the sums!

The Fourier transform

Since the Fourier transform raises weights by *n and* we already know that the cohomology sheaves vary smoothly, we deduce that $H^i_c(\mathbb{A}^n, \mathcal{L}_f)$ is pure of weight i: $w_{i,j} = i$ for all i.

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But the fact that $\mathcal{L}_f$ is a single sheaf does not imply the same thing for the Fourier transform.

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Fourier transform preserves semiperversity!
Since the cohomology groups vary smoothly, their dimension can't jump up, so $H^i_c(\mathbb{A}^n, \mathcal{L}_f) = 0$ for $i > n$.

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Since the Fourier transform preserves perversity, the complex parameterizing $H^i_c(\mathbb{A}^n, \mathcal{L}_f)$ is also perverse.
In particular, its rank can only decrease under specialization. So if f degenerates to g, $\dim(\mathbb{A}^n, \mathcal{L}_f) \geq \dim(\mathbb{A}^n, \mathcal{L}_g)$.

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Using:

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Formula for the rank

$$d_n = (-1)^n + d\frac{(d'-1)^n - (-1)^n}{d'} + (-1)^n(d-d')\chi$$

where

$$\chi := \sum_{\substack{I\subseteq\{1,\dots,r\}\\ 1\leq|I|\leq n-1}} (-1)^{|I|-1}\chi(n-1;e_I) - \sum_{\substack{I\subseteq\{1,\dots,r\}\\ 1\leq|I|\leq n-2}} (-1)^{|I|-1}\chi(n-1;d',e_I)$$

and e_I stands for $e_{i_1},\dots,e_{i_j}$ if $I=\{i_1,\dots,i_j\}$.

Two dimensional case

$$f = f_d + f_{d'} + h \in k[x, y]$$

with $f_{d'}$ squarefree and $\gcd(f_d, f_{d'}) = 1$, then

$$\left| \sum_{(x,y) \in k^2} \psi(f(x, y)) \right| \leq (1 + d(d' - 2) + r(d - d'))q$$