Topology of singularities of holomorphic foliations in $\mathbb{C}^2$

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Introduction

$\mathcal{F}$: germ of holomorphic foliation in $(\mathbb{C}^2, 0)$ defined by

$\omega = a(x,y)dx + b(x,y)dy$ or $X = b(x,y)\partial_x - a(x,y)\partial_y$,

with $a(0,0) = b(0,0) = 0$.

Taylor expansion:

$$\omega = \omega_\nu + \omega_{\nu+1} + \cdots \text{ and } X = X_\nu + X_{\nu+1} + \cdots,$$

with ν =vanishing order at the origin.

If $\nu = 1$ then

$$X_1 = (a_{11}x + a_{12}y)\partial_x + (a_{21}x + a_{22}y)\partial_y$$

determines the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ whose quotient of eigenvalues λ is an analytic invariant of the singularity which is called Camacho-Sad residue.

Case $\nu = 1$

If $\omega = \omega_1 + \cdots$ is generic then $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Poincaré: If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then

$$\omega \sim_{\text{an}} \omega_1 = xdy - \lambda ydx.$$

Guckenheimer: If $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ then

$$xdy - \lambda ydx \sim_{\text{top}} xdy - \mu ydx.$$

($\exists \alpha, \beta \in \mathbb{C},\ \Phi(x,y) = (x|x|^\alpha, y|y|^\beta)$ is a conjugation.)

Remark: After blowing up the origin, $y = tx$, $\widetilde{\Phi}(x,t) = (x|x|^\alpha, t|t|^\beta|x|^{\beta-\alpha})$ and $\widetilde{\Phi}(0,t)$ is constant.

Moduli space: $\mathcal{M}(\mathcal{F}) := \{\mathcal{F}' \sim_{\text{top}} \mathcal{F}\}/\sim_{\text{an}} = \mathbb{H} \ni \lambda$.

Case $\nu > 1$

Seidenberg: There exist a composition π of blow-ups

$$(\mathbb{C}^2, 0) =: (M_0, D_0) \leftarrow (M_1, D_1) \leftarrow \cdots \leftarrow (M_k, D_k)$$

such that all the singularities of $\widetilde{\mathcal{F}} := \pi^* \mathcal{F}$ have

$$\nu = 1 \quad \text{and} \quad \lambda \in \mathbb{C} \setminus \mathbb{Q}_+^*.$$

If ω is generic (in the set $\{\omega_\nu + \cdots\}$) then $k = 1$ and

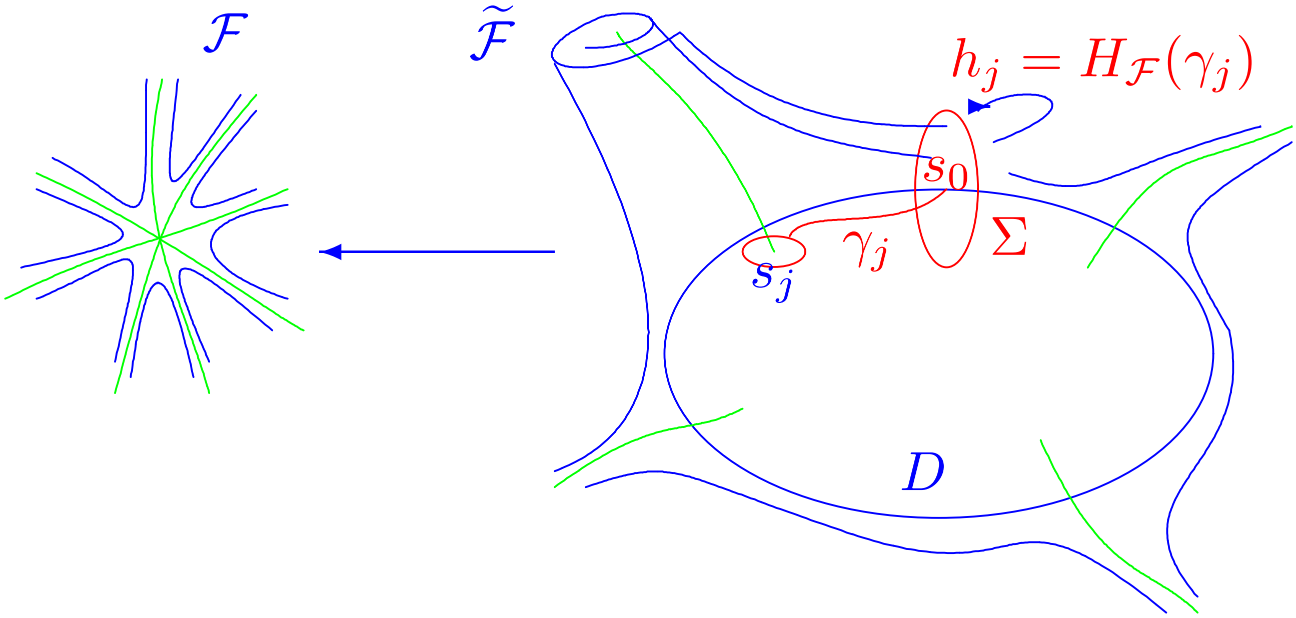
$$P_{\nu+1} := \omega_\nu(x\partial_x + y\partial_y) = \prod_{j=1}^{\nu+1} \ell_j, \quad \ell_i^{-1}(0) \neq \ell_j^{-1}(0).$$

Moreover, the singularities $s_1, \ldots, s_{\nu+1} \in D := D_1$ of $\widetilde{\mathcal{F}}$ have residues $\lambda_1, \ldots, \lambda_{\nu+1}$ given by

$$\frac{\omega_\nu}{P_{\nu+1}} = \sum_{j=1}^{\nu+1} \lambda_j \frac{d\ell_j}{\ell_j},$$

which are generically in $\mathbb{C} \setminus \mathbb{R}$ and generate an additive dense subgroup of $\mathbb{C}$, if $\nu \geq 3$.

Description of the case $k = 1$



Projective holonomy representation:

$$\begin{array}{rcl} H_{\mathcal{F}} : \pi_1(D^*, s_0) & \to & \mathrm{Diff}(\Sigma, s_0) \cong \mathrm{Diff}(\mathbb{C}, 0) \\ \gamma_j & \mapsto & h_j : \zeta \mapsto e^{2i\pi\lambda_j}\zeta + \cdots \end{array}$$

where $s_0 \in D^* := D \setminus \{s_1, \ldots, s_{\nu+1}\}$.

Remark: The holonomy group of $\mathcal{F}$,

$$G := \mathrm{Im}(H_{\mathcal{F}}) \subset \mathrm{Diff}(\mathbb{C}, 0)$$

is well defined up to conjugation and it is generically non abelian, if $\nu \geq 2$.

Analytic classification of the case $k = 1$

Mattei-Moussu, Cerveau-Sad: $\mathcal{F} \sim_{\text{an}} \mathcal{F}'$ if and only if have the same analytic invariants:

- analytic class of the separatrices (including the relative position of the singularities $s_j \in D$);
- residues λ_j at the corresponding singularities;
- projective holonomy representations, i.e. $\exists \psi : (\Sigma, s_0) \to (\Sigma', s_0)$ biholomorphism such that

$$H_{\mathcal{F}'}(\gamma_j) = \psi_*(H_{\mathcal{F}}(\gamma_j)) := \psi \circ H_{\mathcal{F}}(\gamma_j) \circ \psi^{-1}.$$

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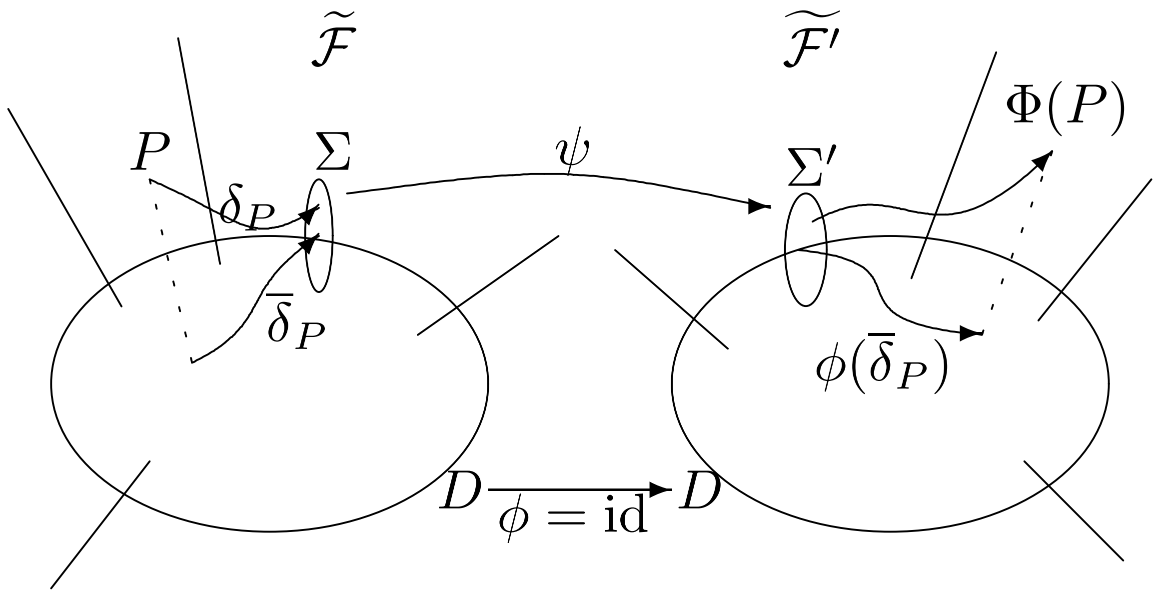
Idea of the proof: Lifting path method.



Topological classification of the case $k = 1$

Cerveau-Sad: $\omega = \omega_\nu + \cdots$ is N.A.G. if

- the holonomy group G is non abelian;
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Cerveau-Sad, Marín: Assume that $\mathcal{F}$ is N.A.G. then $\mathcal{F} \sim_{\text{top}} \mathcal{F}'$ if and only if

- the projective holonomy representations are topologically conjugated, i.e. $\exists \psi : (\Sigma, s_0) \to (\Sigma', s_0')$ biholomorphism and $\exists \phi : (D^*, s_0) \to (D'^*, s_0')$ homeomorphism such that the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(D^*, s_0) & \xrightarrow{\phi_*} & \pi_1(D'^*, s_0') \\ \downarrow H_{\mathcal{F}} & & \downarrow H_{\mathcal{F}'} \\ \mathrm{Diff}(\Sigma, s_0) & \xrightarrow{\psi_*} & \mathrm{Diff}(\Sigma', s_0') \end{array}$$

- the residues λ_j coincide at the singularities corresponding by ϕ.

Moduli spaces of NAG foliations

Corollary: If $\nu = 2$ then the moduli space

$$\mathcal{M}(\omega) = \{\omega' \sim_{\text{top}} \omega\}/\sim_{\text{an}}$$

of $\omega = \omega_2 + \cdots$ NAG is trivial.

Corollary: If $\nu = 3$ and $\omega = \omega_3 + \cdots$ is NAG then $\mathcal{M}(\omega)$ is a connected covering (generically simply connected) of the configuration space

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Corollary: If $\nu > 3$ and $\omega = \omega_\nu + \cdots$ is NAG then $\mathcal{M}(\omega)$ contains a connected covering of the configuration space $F_{3,\nu-2}(\mathbb{S}^2)$ whose fundamental group is completely determined by the projective holonomy representation of ω. Moreover, it is generically trivial if and only if the Gassner representation of the pure braid group is faithful.

Idea of the proof ([$\Rightarrow$])

Let $\Phi : U \to U'$ be a homeomorphism conjugating the foliations $\mathcal{F}$ and $\mathcal{F}'$ with separatrices S and S'. Let $U^* = U \setminus S$ and $U'^* = U' \setminus S'$. Then $\Phi_* : \pi_1(U^*) \to \pi_1(U'^*)$ is an isomorphism.

Since $\pi_1(U^*) \cong \pi_1(D^*) \oplus \pi_1(\Sigma^*)$, we can pass Φ_* to the quotient by the center and we obtain an algebraic isomorphism $\varphi : \pi_1(D^*) \to \pi_1(D'^*)$ which verifies the algebraic characterisation of Nielsen theorem in order to exist a homeomorphism $\phi : D^* \to D'^*$ inducing φ in homotopy.

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Let $c = [\partial\Sigma]$ and $c' = [\partial\Sigma']$ be the generators of the center. Since $\Phi_*(c) = c'$ we can modify tangentially Φ in order to that $\Phi(C) = C' \subset \Sigma'$. Using contractive holonomies we can extend $\Phi_{|C}$ to a homeomorphism $\psi : (\Sigma, s_0) \to (\Sigma', s_0')$.

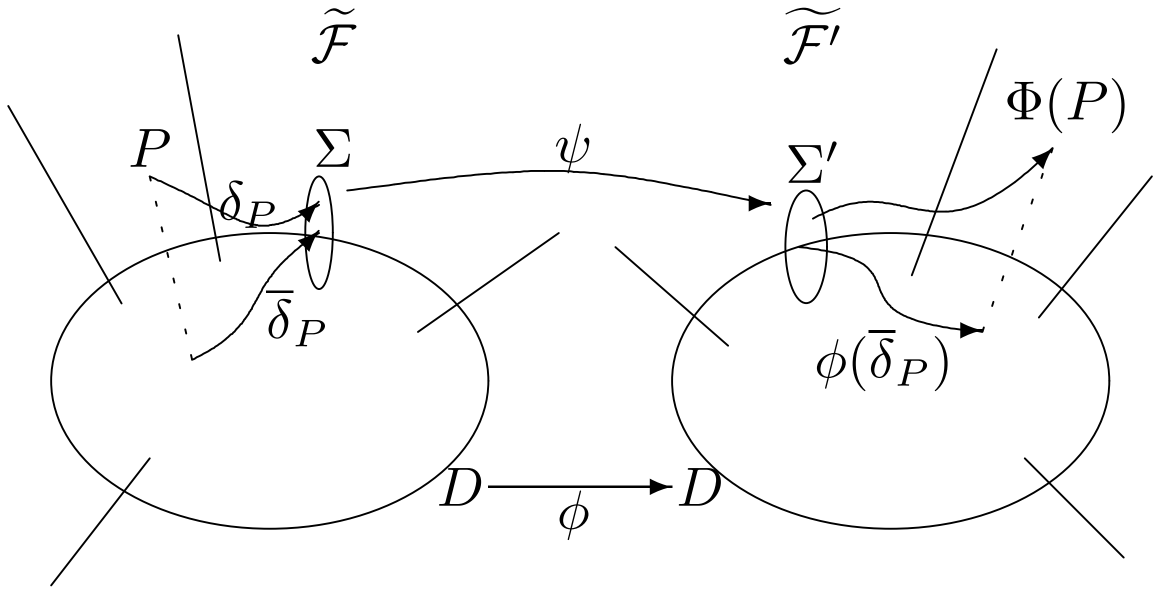
Hypothesis N.A.G. imply the rigidity of the holonomy group G, i.e. in fact ψ is a biholomorphism.

Using the transverse Hopf fibration, we can see that

$$\psi_* \circ H_{\mathcal{F}} = H_{\mathcal{F}'} \circ \phi_*.$$

Idea of the proof ([⇐])

We construct $\Phi : U \to U'$ using the lifting path method from the data $\phi : D \to D'$ and $\psi : (\Sigma, s_0) \to (\Sigma', s'_0)$.



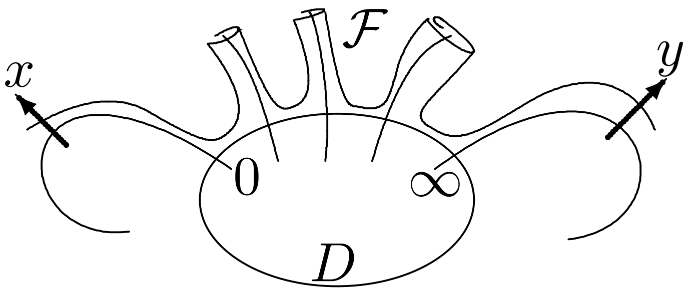
<u>Remark:</u> We can assume that ϕ is holomorphic near each singularity $s_j \in D$.

Generalization to TQH germs

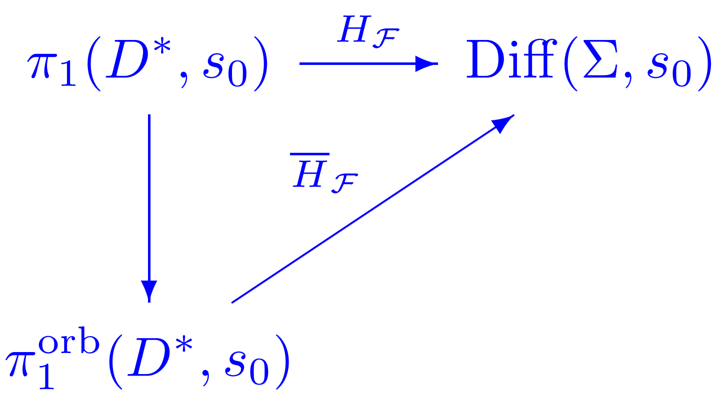
Definition: $\mathcal{F}$ is TQH if its separatrix set S is topologically conjugated to a QH curve, i.e. given by a quasi-homogeneous polynomial $\sum_{\alpha i+\beta j=\delta} a_{ij}x^i y^j = 0$.

Example: If $\omega = \omega_1 + \cdots$ is generic in the set $\{\omega :\ \omega_1 = 2ydy\}$ then $S \sim_{\text{an}} \{y^2 - x^3 = 0\}$.

Property: If $\mathcal{F}(= \mathcal{F}_{2ydy+\dots})$ is a generic TQH germ then the desingularization of $S(= \{y^2 - x^3 = 0\})$ is given by



and $\pi_1(U^*)(= \langle x, y \,|\, y^2 = x^3 =: c\rangle)$ has center $\langle c\rangle \cong \mathbb{Z}$ and $\pi_1(U^*)/\langle c\rangle \cong \pi_1^{\text{orb}}(D^*)$. Moreover, the projective holonomy representation $H_{\mathcal{F}}$ of D factorizes as follows:



Topological classification of TQH germs

Definition: A TQH germ $\mathcal{F}$ is NAG if and only if

- the residues of the singularities in $D \setminus \{0, \infty\}$ are non real and generate a dense additive subgroup of $\mathbb{C}$;
- the holonomy group G of D is non abelian.

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- the holonomy group G of D is non abelian.

<u>Theorem:</u> Assume that $\mathcal{F}$ is a NAG TQH germ.
Then $\mathcal{F} \sim_{\text{top}} \mathcal{F}'$ if and only if

- the projective holonomy representations are topologically conjugated, i.e.
  $\exists \psi : (\Sigma, s_0) \to (\Sigma', s_0')$ biholomorphism and
  $\exists \varphi : \pi_1^{\text{orb}}(D^*, s_0) \to \pi_1^{\text{orb}}(D'^*, s_0')$ geometric isomorphism such that the following diagram is commutative:

$$\begin{array}{ccc} \pi_1^{\text{orb}}(D^*, s_0) & \xrightarrow{\varphi} & \pi_1^{\text{orb}}(D'^*, s_0') \\ \Big\downarrow{\scriptstyle \overline{H}_{\mathcal{F}}} & & \Big\downarrow{\scriptstyle \overline{H}_{\mathcal{F}'}} \\ \text{Diff}(\Sigma, s_0) & \xrightarrow{\psi_*} & \text{Diff}(\Sigma', s_0') \end{array}$$

- the residues of the singularities corresponding by φ coincide.

Study of the general case

Example: If the separatrices of $\mathcal{F}$ are $S = \{(y^2 - x^3)(x^2 - y^3) = 0\}$ then $\Gamma := \pi_1(U^*)$ can be presented as

$$\Gamma = \langle a_1, c_1, d_1, b, d_2, c_2, a_2 \mid c_1 = a_1^2, a_1 d_1 b = c_1, \\ c_1 c_2 = b^4, b d_2 a_2 = c_2, a_2^2 = c_2, \ldots \rangle,$$

where $\ldots$ are the commutation relations: $[a_1, c_1] = [c_1, d_1] = [c_1, b] = [b, c_2] = [c_2, d_2] = [c_2, a_2] = 1$. It can be checked that its center is trivial.

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However, $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$, where

$$\begin{aligned} \Gamma_0 &= \langle b, c_1 \rangle = \langle b, c_2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \\ \Gamma_1 &= \langle a_1, c_1, d_1, b \rangle \\ \Gamma_2 &= \langle a_2, c_2, d_2, b \rangle \end{aligned}$$

The center of Γ_i is $\langle c_i \rangle$ for $i = 1, 2$.

In fact, the 3-manifold $M^3 := S^3_\epsilon \setminus V(S)$ admits an embedded 2-torus $T_0 \subset M$ such that $\pi_1(T_0) = \Gamma_0$ and $M \setminus V(T_0) = M_1 \sqcup M_2$, where M_i are Seifert manifolds with $\pi_1(M_i) = \Gamma_i$ for $i = 1, 2$.

Jaco-Shalen-Johannson decomposition

Jaco-Shalen,Johannson: Let M be an irreducible, ∂-irreducible Haken 3-manifold. Then there exists a minimal system $\mathcal{T}$ of disjoint incompresible tori such that $M \setminus \mathcal{T} = \sqcup M_i$, where each M_i is either Seifert or simple. Moreover, $\mathcal{T}$ is unique up to isotopy.

Remark: For each torus $T \subset \mathcal{T}$ and each piece M_i adjancent to T the canonical inclussions induce monomorphisms:

$$\pi_1(T) \hookrightarrow \pi_1(M_i \cup T) \cong \pi_1(M_i) \hookrightarrow \pi_1(M).$$

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Neumann,Popescu: Let $\mathcal{F}$ be a germ of a singular holomorphic foliation in $(\mathbb{C}^2, 0)$ with separatrix set S. For $\epsilon > 0$ small enough, the 3-manifolds $M := S^3_\epsilon \setminus V(S)$ are all diffeomorphic and its JSJ decomposition is characterized by the fact that each piece M_i is associated to a divisor D_i of valence ≥ 3 in the desingularization of S and admits a Seifert fibration induced by the Hopf fibration of D_i.

Remark: We can think this theorem as a TQH decomposition of $U^* = \cup Q_i$.

Cerveau-Sad conjecture

Conjecture: The projective holonomy representations of a germ of generalized curve $\mathcal{F}$ are topological invariants.

Partial answer: Is true in the NAG TQH case.

Remark: It is sufficient to prove the conjecture for the projective holonomy representations associated to the divisors D_i of valence ≥ 3.

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Program: Use the JSJ decomposition in order to apply the precedent techniques to the general (but generic) case.

Let $\Phi : U \to U'$ be a homeomorphism conjugating $\mathcal{F}$ and $\mathcal{F}'$. The unicity (up to isotopy) of the JSJ decomposition of $M' = S^3_\epsilon \setminus V(S')$ implies that $\Phi_* : \pi_1(U^*) \to \pi_1(U'^*)$ maps $\Gamma_i := \pi_1(M_i) \subset \pi_1(U^*)$ onto $\Gamma'_i := \pi_1(M'_i) \subset \pi_1(U'^*)$. Therefore the center $\langle c_i \rangle \cong \mathbb{Z}$ of Γ_i is maped by Φ_* onto the center of $\langle c'_i \rangle$ of Γ'_i. We can define geometric isomorphisms

$$\varphi_i : \pi_1^{\mathrm{orb}}(D_i^*) \cong \Gamma_i / \langle c_i \rangle \to \Gamma'_i / \langle c'_i \rangle \cong \pi_1^{\mathrm{orb}}(D_i'^*)$$

and under suitable hypothesis we could also construct homeomorphisms $\psi_i : \Sigma_i \to \Sigma'_i$ between transverse fibres through D_i and D'_i.

Foliated 1-connexity I

Problem: To assure that the diagram is commutative:

$$\begin{array}{ccc} \pi_1^{\mathrm{orb}}(D_i^*, s_i) & \xrightarrow{\varphi_i} & \pi_1^{\mathrm{orb}}(D_i'^*, s_i') \\ \downarrow{\scriptstyle \overline{H_i}} & & \downarrow{\scriptstyle \overline{H_i'}} \\ \mathrm{Diff}(\Sigma, s_0) & \xrightarrow{(\psi_i)_*} & \mathrm{Diff}(\Sigma_i', s_i') \end{array}$$

Main difficulty: We known that if $\gamma \subset L \cap Q_i$ then $\gamma' = \Phi(\gamma) \subset L'$ is homotopic (in U'^*) to a path $\beta' \subset Q_i'$. However, we need a foliated homotopy between γ' and another path $\alpha' \subset L' \cap Q_i'$.

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Definition of foliated 1-connexity of Q in U^*:

$Q \hookrightarrow_{\mathcal{F}} U^*$ if and only if for each γ contained in a leaf L of $\mathcal{F}$ and $\beta \subset Q$ homotopic in U^* with the same endpoints, there exists $\alpha \subset L \cap Q$ homotopic to β in Q and homotopic to γ in L:

$$\begin{array}{ccc} \exists \alpha \subset Q \cap L & \longrightarrow & L \supset \gamma \\ \Big\downarrow & & \Big\downarrow \\ \beta \subset Q & \longrightarrow & U^* \supset \beta \sim \gamma \end{array}$$

Foliated 1-connexity II

Equivalent description:

Let $\mathcal{F}$ be a foliation in M and $N \subset M$. Let $\mathcal{C}_N$ be the set of continuous paths in N and $\approx_N$ the homotopy relation in N.

We denote

$$[\mathcal{C}_N] := (\mathcal{C}_N / \approx_N), \quad [\mathcal{C}_\mathcal{F}] := \bigsqcup_{L \in \mathcal{F}} (\mathcal{C}_L / \approx_L)$$

and

$$[\mathcal{C}_{\mathcal{F}_{|N}}] := \bigsqcup_{L \in \mathcal{F}_{|N}} (\mathcal{C}_L / \approx_L).$$

Definition: N is 1-$\mathcal{F}$ connected in M if and only if the following sequence is exact:

$$[\mathcal{C}_{\mathcal{F}_{|N}}] \xrightarrow{\ u\ } [\mathcal{C}_N] \times [\mathcal{C}_{\mathcal{F}_{|M}}] \xrightarrow{\ v\ } [\mathcal{C}_M]$$

where

$$u([\alpha]_{L_N}) = ([\alpha]_N, [\alpha]_{L_M})$$

and

$$v([\beta]_N, [\gamma]_{L_M}) = [\beta\gamma^{-1}]_M.$$

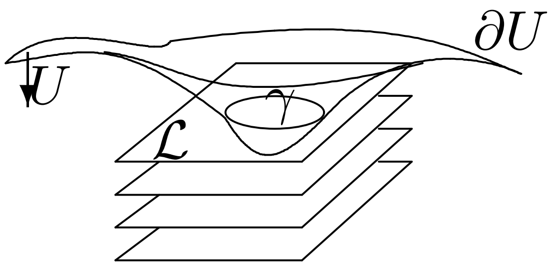
Foliated 1-connexity III

Immediate properties:

- reflexive: $U \looparrowright_{\mathcal{F}} U$;
- transitive: $W \looparrowright_{\mathcal{F}} V$, $V \looparrowright_{\mathcal{F}} U \Longrightarrow W \looparrowright_{\mathcal{F}} U$;
- characterization of the incompresibility of the leaves:

$$\{p\} \looparrowright_{\mathcal{F}} U \quad \Longleftrightarrow \quad \pi_1(\mathcal{L}_p) \hookrightarrow \pi_1(U).$$

Remark: The foliated 1-connexity depends on the shape of U:



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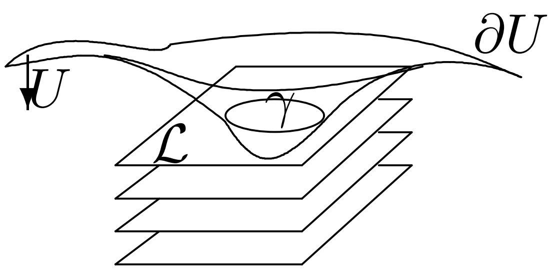

Remark: If $\mathcal{F}$ admits a holomorphic first integral $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ (with $S = f^{-1}(0)$), then for $\varepsilon >> \eta > 0$ small enough the restriction of f to $U^*_{\varepsilon,\eta} := \mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}^*_\eta)$ is a locally trivial fibration by Milnor theorem. Consequently, we have the long exact sequence in homotopy which ends with

$$1 \to \pi_1(F) \to \pi_1(U^*_{\varepsilon,\eta}) \xrightarrow{f_*} \pi_1(\mathbb{D}^*_\eta) \to 1,$$

where F is a fibre of f, i.e. a leaf of $\mathcal{F}$.

Foliated 1-connexity IV

Theorem (Mattei-Marín): Let $\mathcal{F}$ be a generalized curve such that all the singularities of $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ are linealizable. Then there exists a system of neighborhoods $\mathcal{U}$ of the origin such that, for any $U \in \mathcal{U}$ and each arbitrary collection $Q_1, \ldots, Q_k$ of TQH pieces of U^* we have

$$Q \looparrowright_{\mathcal{F}} U^* \quad \text{with} \quad Q = \bigcup_{i=1}^{k} Q_i.$$

Notation: If $k = 0$ then $Q := \{p\}$.

Corollary: The leaves of $\mathcal{F}$ are incompresible in U^*, i.e. the natural inclusion of a leaf $L \ni p$ into U^* induces a monomorphism

$$\pi_1(L, p) \hookrightarrow \pi_1(U^*, p).$$