Transcendental complete trajectories of polynomial vector fields on $\mathbb{C}^2$.

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• A **vector field** X on $\mathbb{C}^2$ is a section of the tangent bundle of $\mathbb{C}^2$

$$X = R\frac{\partial}{\partial x} + S\frac{\partial}{\partial y}, \quad R, S \in \mathcal{O}_{\mathbb{C}^2}.$$

Associated to X we have the following system:

$$\begin{cases} \dfrac{dx}{dt} = R(x,y) \\ \\ \dfrac{dy}{dt} = S(x,y) \end{cases}$$

Existence and uniqueness of a local solution $\Rightarrow$ for all $z \in \mathbb{C}^2$ there exists a local solution $t \in \mathbb{D}_{r_z} \mapsto \varphi_z(t)$ of X through z.

• Given the local solution $t \mapsto \varphi_z(t)$ of X through $z \in \mathbb{C}^2$, we can extend it by analytic continuation along the paths in $\mathbb{C}$ to the maximal domain of definition Ω_z in the sense of Riemann. This map

$$\varphi_z : \Omega_z \to \mathbb{C}^2$$

is called the **solution** of X through z and its image C_z defines the **trajectory**.

• Suppose that X is **polynomial** and of degree m. Let us consider the atlas

$$\{(U_i, \phi_i^{-1})\}_{i=0,1,2}$$

of $\mathbb{CP}^2$ given by the three basic charts

$$U_i := \{[z_0 : z_1 : z_2], z_i \neq 0\} \quad \text{para} \quad i = 0, 1, 2;$$

and the homeomorphisms $\phi : \mathbb{C}^2 \to U_i$ defined by

$$\phi_0(z_1, z_2) = [1 : z_1 : z_2], \quad \phi_1(y_1, y_2) = [y_1 : 1 : y_2],$$

$$\phi_2(w_1, w_2) = [w_1 : w_2 : 1].$$

• X defines a **rational vector field** on $\mathbb{CP}^2$ which is given in each chart (U_i, ϕ_i^{-1}) by $(\phi_i^{-1} \circ \phi_0)_* X$. The **pole** of X along L_∞ is of order $d = m - 1$ order $m - 2$.

• If we remove the pole of X, we obtain in each cahrt (U_i, ϕ_i^{-1}) a polynomial X_i with isolated zeroes. Since in each of the intersections $U_i \cap U_j$ (with $i \neq j$)

$$X_i = \xi_{ij} \cdot X_j, \;\; \text{donde} \;\; \xi_{ij} \in \mathcal{O}^*(U_i \cap U_j),$$

the fields $\{X_i\}_{i=0,1,2}$ define a global section $\mathcal{F}_X$ of

$$T\mathbb{CP}^2 \otimes \mathcal{O}(d)$$

which is the **foliation defined by** X.

• X is **complete** if $\forall\ z \in \mathbb{C}^2$ the solution $\varphi_z : \mathbb{C} \to C_z$ of X through z is entire.

• *The complete trajectories of X are biholomorphic either to $\mathbb{C}$, o or to a punctured plane $\mathbb{C}^*$.*

If $\varphi_z : \mathbb{C} \to C_z$ is entire $\Rightarrow$

$C_z \simeq \mathbb{C}/\Gamma$ with Γ a lattice of $(\mathbb{Z} \times \mathbb{Z}, +) \Rightarrow$

the range of Γ is 0 or 1 ($\mathbb{C}^2$ is Stein).

• A trajectory C_z of a holomorphic vector field X on $\mathbb{C}^2$ is **proper** if $\overline{C}_z$ (topological closure in $\mathbb{C}^2$) defines an analytic curve.

• If C_z is proper, $\overline{C}_z \setminus C_z$ is at most a finite set of points.

Example. Any trajectory of type $\mathbb{C}^*$ of a complete vector field in $\mathbb{C}^2$ is proper (M. Suzuki).

• If X is polynomial, since C_z is contained in a leaf of $\mathcal{F}_X \Rightarrow$ we can define its **limit set**, $lim\,(C_z)$, as the boundary of C_z in $\mathbb{CP}^2$.

• When C_z is proper there exist two possibilities:

⋆ $lim\,(C_z) \cap L_\infty = \{\text{ finite set of points }\} \Rightarrow \overline{C}_z$ is algebraic.

⋆ $lim\,(C_z) \cap L_\infty = L_\infty$.

• **Transcendental** means proper and non algebraic.

• A polynomial vector field on $\mathbb{C}^2$ is determined by a transcendental trajectory C_z (up to multiplication by a constant). Thus we can ask *if the fact that X has a complete transcendental trajectory C_z implies that X is complete.*

Theorem. *Any polynomial vector field on $\mathbb{C}^2$ with a complete transcendental trajectory is complete.*

Sketch of the proof.

1.- Previous work. In a recent paper:

A. Bustinduy, *On the entire solutions of a polynomial vector field on* $\mathbb{C}^2$, Indiana University Math. Journal vol. 53 (2), 2004,

we have provided a classification of polynomial fields with a complete transcendental trajectory:

Theorem. *Let C_z be a complete transcendental trajectory of X:*

***A) If** C_z **is of type** $\mathbb{C}^*$, up to a polynomial automorphism, we have two possibilities:*

***i) There is a line invariant by** X, and X is one of the following vector fields:*

i.1)

$$\lambda x \frac{\partial}{\partial x} + [a(x)y + b(x)] \frac{\partial}{\partial y},$$

where $a(x)$, $b(x) \in \mathbb{C}[x]$ and $\lambda \in \mathbb{C}^*$.

i.2)

$$x[n\, f(x^m y^n) + \alpha] \frac{\partial}{\partial x} - y[m\, f(x^m y^n) + \beta] \frac{\partial}{\partial y},$$

with $m, n \in \mathbb{N}^*$, $f(z) \in z \cdot \mathbb{C}[z]$, $\alpha, \beta \in \mathbb{C}$ such that $\beta/\alpha \in \mathbb{Q}^*$ and $\alpha m - \beta n \in \mathbb{C}^*$.

i.3)

$$x[n\, S + \alpha] \frac{\partial}{\partial x} + \left\{ -\frac{[nT + m(x^\ell y + p(x))]\, S + \alpha T}{x^\ell} \right\} \frac{\partial}{\partial y},$$

where $m, n, \ell \in \mathbb{N}^*$, $\alpha \in \mathbb{C}^*$, $p(x) \in \mathbb{C}[x]$ of degree $< \ell$ with $p(0) \neq 0$, $T = \ell x^\ell y + x p'(x)$, $S = f(x^m (x^\ell y + p(x))^n)$ where $f(z) \in z \cdot \mathbb{C}[z]$, and

$$[n\, x p'(x) + m\, p(x)] S + \alpha x p'(x) \in x^\ell \cdot \mathbb{C}[x, y].$$

ii) There is no a line invariant by X*,* $\mathcal{F}_X$ *is* P*-complete with* P *of type* $\mathbb{C}^*$*, and after rational change of coordinates* H*, it can be reduced to a normal form:*

$$H^*X = u^k \cdot \left\{ a(v)u\frac{\partial}{\partial u} + c(v)\frac{\partial}{\partial v} \right\}, \tag{1}$$

where $k \in \mathbb{Z}$*,* $a(v) \in \mathbb{C}[v]$*, and* $c(v) = \mu\, v^j (v^n - s)^r$*, with* $j \in \mathbb{N}$*,* $n, r \in \mathbb{N}^*$*,* s*,* $\mu \in \mathbb{C}^*$*.*

B) If C_z ***is of type*** $\mathbb{C}$*, there exists an analytic automorphism* Φ *of* $\mathbb{C}^2$ *such that*

$$\Phi^*X = f(y)\frac{\partial}{\partial x},$$

with f *a non-vanishing entire function.*

• We reduce the proof of this theorem to see that for a polynomial vector field with a complete transcendental trajectory C_z of type $\mathbb{C}^*$ the case (ii) in Theorem can not occur.

2.- Analyze P-complete foliations with P of type $\mathbb{C}^*$.

• There are many distinct polynomials of type $\mathbb{C}^*$ after a polynomial automorphism, that can be written as:

$$P = x^m (x^\ell y + p(x))^n, \text{ where } m, n \in \mathbb{N}^*,\ \ell \in \mathbb{N}$$

and $p \in \mathbb{C}[x]$ of degree $< \ell$ such that $p(0) \neq 0$ if $\ell > 0$, or $p \equiv 0$ if $\ell = 0$ (Suzuki-Saito's Theorem).

• The rational map H is defined according to the relations

$$x = u^n \quad \text{and} \quad x^\ell y + p(x) = v\, u^{-m}$$

so that the inverse image of a given fiber $P^{-1}(k)$, with $k \neq 0$, is defined by the family of the n punctured lines

$$v = \xi_{1k} \setminus \{(0, \xi_{1k})\}, \dots, v = \xi_{nk} \setminus \{(0, \xi_{nk})\},$$

where ξ_{jk} with $j = 1, \dots, n$ are the n–th roots of k. This way $H^* \mathcal{F}_X$ is in fact a foliation v–complete that leaves invariant the line $\{u = 0\}$.

3.- Use H to parametrize the two ends of C_z.

• $C_z = \Sigma\,(\text{transcendental end}) \cup \Delta\,(\text{algebraic end})$.

• Given a point $(x_0, y_0) \in \Sigma$, we can ask ourselves if Σ can be explicitly parametrizated around (x_0, y_0).

• We can suppose that (x_0, y_0) is not in $P^{-1}(0)$ (singular fiber): P is of type $\mathbb{C}^* \Rightarrow \Sigma$ is at infinity with respect to $P \Rightarrow$ $\Sigma \cap \{P = 0\} = \emptyset$.

• Let us consider $(u_0, v_0) \in H^{-1}(x_0, y_0)$, that is,

$$u_0 = \sqrt[n]{x_0} \quad \text{and} \quad v_0 = \left(\sqrt[n]{x_0}\right)^m (x_0^\ell y_0 + p(x_0)),$$

and the leaf $\mathcal{C}$ of $H^*\mathcal{F}_X$ which contains it.

• Instead of

$$Y = a(v)u\frac{\partial}{\partial u} + c(v)\frac{\partial}{\partial v},$$

we can take a convenient multiple of it,

$$(1/c(v)) \cdot Y,$$

that we can integrate.

Lemma. Given (x_0, y_0) in Σ, $\forall$ $(u_0, v_0) \in H^{-1}(x_0, y_0)$, there exists a disk $\mathbb{D}_r(v_0)$ such that the map $\gamma : \mathbb{D}_r(v_0) \to \Sigma$ given by

$$\gamma(t) = H\left(u_0\, e^{\int_{v_0}^{t} \frac{a(z)}{c(z)}\, dz}, t\right) \quad (*)$$

parametrizes a neighborhood of (x_0, y_0) in Σ.

• Let us take a disk $\mathbb{D}_R$ containing $c(v) = 0$ and (x_0, y_0) in Σ such that $(u_0, v_0) \in H^{-1}(x_0, y_0)$ verifies $v_0 \in \mathbb{C} \setminus \mathbb{D}_R$, then $(*)$ can be extended by analytic continuation along paths in $\mathbb{C} \setminus \mathbb{D}_R$ from v_0 as a multivaluated holomorphic function. If we now take the biholomorphism

$$p : \overline{\mathbb{D}}_{1/R} \setminus \{0\} \to \mathbb{C} \setminus \mathbb{D}_R, \quad s \quad \mapsto \quad 1/s = t$$

Definition. Given a point (x_0, y_0) in Σ, for each $(u_0, v_0) \in H^{-1}(x_0, y_0)$ there exists $R > 0$ such that

$$\bar{\gamma} := \gamma \circ p = H\left(u_0\, e^{\int_{s_0}^{s} -\frac{a(1/z)}{c(1/z)} \frac{dz}{z^2}}, 1/s\right), \quad \text{with } s_0 = 1/v_0,$$

defines a multivaluated map from $\overline{\mathbb{D}}_{1/R} \setminus \{0\}$ to Σ, that we will call the **multivaluated parametrization of** Σ.

• **Normal form for the parametrization of Σ.**

If we analyze the normal forms of the 1-form $\omega = a(z)/c(z)\, dz$ at poles and zeroes we obtain:

Proposition. Given the parametrization $\bar{\gamma}$ of Σ, there is a biholomorphism $\phi : V \subset \mathbb{C} \to \mathbb{D}_{1/R}$ with $\phi(0) = 0$ such that $\bar{\gamma} \circ \phi(w)$ is equal to

1.-
$$H\left(u_0\, e^{-\frac{{w_0}^{l+1}}{l+1}}\, e^{\frac{w^{l+1}}{l+1}}, 1/\phi(w)\right),$$
if ω has a zero of order $l \geq 0$ at ∞,

2.-
$$H\left(u_0\, e^{-\lambda \log w_0}\, e^{\lambda \log w}, 1/\phi(w)\right),$$
if ω has a pole of order $l = 1$ at ∞, and

3.-
$$H\left(u_0\, e^{(-\frac{{w_0}^{-l+1}}{-l+1} - \lambda \log w_0)}\, e^{(\frac{w^{-l+1}}{-l+1} + \lambda \log w)}, 1/\phi(w)\right),$$
if ω has a pole of order $l \geq 2$ at ∞,

with $\phi(w) = s$ and $\phi(w_0) = s_0$. These equations are the normal forms of the parametrization of Σ.

• The same ideas work to parametrize the algebraic end Δ and obtain its normal forms:

• We can take $v_0 \in \mathbb{D}_{r_1}(\xi_{1s})$ with $(x_0, y_0) = H(u_0, v_0)$, obtain a local solution δ of $(1/c(v)) \cdot Y$ and extend it by analytic continuation along paths in $\mathbb{D}_{r_1}(\xi_{1s})$ from v_0 as a multivaluated holomorphic function.

4.- The poles of $\omega = a(z)/c(z)\,dz$ including the point at infinity are of order one $\Leftrightarrow$ $d\log u = du/u = \omega$ belongs to the Fuchsian class

• The global one-form of times.

$\mathcal{F}_X$ is P-complete with P of type $\mathbb{C}^*$ $\Rightarrow$ if we remove the singular locus of $dP(x,y)$, the one-form

$$\eta = [(m + n\ell)x^{\ell}y + mp(x) + nxp'(x)]dx + nx^{\ell+1}dy$$

is such that

$$\eta(X) = \varrho \cdot x^{\alpha} \cdot (x^{\ell}y + p(x))^{\beta} \cdot [x^m(x^{\ell}y + p(x))^n - s]^{\gamma},$$

for $\varrho \in \mathbb{C}$, α, β and $\gamma \in \mathbb{N}$, and thus

$$\tau = \frac{1}{x^{\alpha} \cdot (x^{\ell}y + p(x))^{\beta} \cdot [x^m(x^{\ell}y + p(x))^n - s]^{\gamma}} \cdot \eta$$

verifies that

$$\tau(X) \equiv 1.$$

• τ has the following property:

for any path $c : [0,1] \to C_z$ from $p = c(0)$ to $q = c(1)$ contained in C_z that avoids the set of poles

$$x^{\alpha} \cdot (x^{\ell} y + p(x))^{\beta} \cdot [x^{m}(x^{\ell} y + p(x))^{n} - s]^{\gamma} = 0$$

of τ it holds that

$$\int_c \tau = \int_0^1 c^* \tau$$

is the complex time (up to periods) required by the flow of X to travel from p to q. We call τ **global one-form of times.**

• This τ along with the normal forms of the parametrizations of Σ and Δ allows to prove:

Lemma. If $du/u = \omega$ does not belong to the Fuchsian class, there exist a sequence $\{(x_q, y_q)\}_{q \in \mathbb{N}}$ of points in C_z which converges to a point ϱ in its boundary and paths $e_q : [0,1] \to \Sigma$ from $(x_0, y_0) = e_q(0)$ to $(x_q, y_q) = e_q(1)$ such that

$$\lim_{q \to \infty} \left\| \int_{e_q} \tau \right\| < \infty.$$

• After this lemma,

$$\omega = \frac{a(z)}{\mu\, z(z^n - s)}\, dz = \frac{a(z)}{\mu\, z \prod_{i=1}^n (z - \xi_{is})}\, dz.$$

• In order that the point at infinity ∞ also has order one the degree of $a(z)$ cannot exceed $n \Rightarrow$ we can obtain a partial **fraction expansion** of ω,

$$\omega = \left\{ \frac{r_0}{z} + \sum_{i=1}^{n} \frac{r_i}{(z - \xi_{is})} \right\} dz, \quad \text{with } r_0, r_i \in \mathbb{C}^*,$$

which gives us a (multivaluated) meromorphic first integral of $H^* \mathcal{F}_X$:

given $(u_0, v_0) \notin \{\mu\, v \prod_{i=1}^n (v - \xi_{is}) = 0\}$, the solution $(u(t), v(t))$ is given by

$$u(t) = u_0\; e^{\int_{v_0}^{t} \frac{a(z)}{\mu\, z \prod_{i=1}^n (z - \xi_{is})}\, dz}$$
$$v(t) = t.$$

• As

$$\int \frac{a(z)}{\mu\, z \prod_{i=1}^{n}(z-\xi_{is})}\, dz = \log\left(z^{\,r_0} \cdot \prod_{i=1}^{n} (z-\xi_{is})^{r_i} \right),$$

$$\begin{aligned} u(t) &= u_0 \left(\frac{t^{\,r_0} \cdot \prod_{i=1}^{n} (t-\xi_{is})^{r_i}}{v_0^{r_0} \cdot \prod_{i=1}^{n} (v_0-\xi_{is})^{r_i}} \right) \\ v(t) &= t. \end{aligned}$$

and

$$f = \frac{u}{v^{\,r_0} \cdot \prod_{i=1}^{n} (v-\xi_{is})^{r_i}}$$

is a **(multivaluated) meromorphic integral of** $H^*\mathcal{F}_X$**.**

• Let us consider C_z and the finite set of points $\mathcal{X}$ in which C_z (maybe) meets $\{x=0\}$.

• As H is a finite covering map from $u \neq 0$ to $x \neq 0$, it is proper $\Rightarrow$ $H^{-1}(C_z \setminus \mathcal{X})$ is a finite disjoint union $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_h$ of parabolic transcendental trajectories of H^*X.

• Let us fix one of them, for instance $\mathcal{L}_1 \Rightarrow \mathcal{L}_1$ is the Riemann surface defined by a fiber of f on a value of $\mathbb{C}^*$.

• Since $T(u,v) = (ab^{-1}u, v)$ transforms $f^{-1}(a)$ in $f^{-1}(b) \Rightarrow$ $H^*\mathcal{F}_X$ is a **parabolic proper foliation** in $\mathbb{C}^2 \Rightarrow$
there exists a (non-constant) single-valued meromorphic first integral of $H^*\mathcal{F}_X$ (M. Suzuki).

• This implies that the numbers r_0 and r_i (for any i) are in $\mathbb{Q}$, which contradicts the fact that C_z is transcendental.