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METRIC FIXED POINT RESULTS CONCERNING MEASURES OF NONCOMPACTNESS

Introduction

Fixed Point Theory has two main branches: on the one hand we can consider the results that are deduced from topological properties and on the other hand those which can be obtained using metric properties.

Among the former the most important result is Brouwer's theorem, which assures that the euclidean unit ball $\overline{B}_n(0,1)$ in $\mathbb{R}^n$ has the fixed point property for continuous functions, that is, if $f : \overline{B}_n(0,1) \to \overline{B}_n(0,1)$ is a continuous mapping, there exists a point $x_0 \in \overline{B}_n(0,1)$ such that $f(x_0) = x_0$.

Schauder [S, 1930] generalized the Brouwer fixed point theorem to infinite dimensional Banach spaces for the case of compact operators. Compactness plays an essential role in the proof of the Schauder fixed point theorem. However, there are some important problems where the operators are not compact.

The first step to extend the Schauder theorem to the setting of noncompact operators was given by G. Darbo [D, 1955]. The main idea is to define a new class of operators which map any bounded set in a "more compact" set. In order to state the property: A set is mapped into a "more compact" set, we need to define some "measure of noncompactness". The first such measure was defined by Kuratowski [Ku1, 1930] in connection with certain problems of General Topology.

Probably the best known and important metric fixed point theorem is the Banach contraction principle, which assures that every contraction from a complete metric space into itself has a unique fixed point. A mapping T from a metric space (X, d) into itself is said to be a contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for every $x, y \in X$. Simple examples show that the Banach theorem does not hold letting k to be equal to 1. Such mappings are called nonexpansive. However in the sixties Browder, Gohde and Kirk proved that under suitable geometric conditions nonexpansive mappings have the fixed point property.

Although historically the two branches of the fixed point theory have had a separated development, in recent years measures of noncompactness have also been utilized to define new geometrical properties of Banach spaces which are interesting for fixed point theory for nonexpansive mappings(see for instance [KMP], [ADL1], [DL2], [Se2], [Pr5] and references therein).

Our goal in this chapter is to study the use of measures of noncompactness in metric fixed point theory. We have divided the chapter in five sections.

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1. Kuratowski and Hausdorff measures of noncompactness

In this section we define the Kuratowski and Hausdorff measures of noncompactness (MNCs) and study their basic properties. As we know, if B is a bounded non precompact set, there is a number $\varepsilon > 0$ such that B cannot be covered by finitely many sets with diameter $\leq \varepsilon$. Hence, we can give the following definition:

DEFINITION 1. *Let (X, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of X. For every $B \in \mathcal{B}$ we define the Kuratowski measure of noncompactness $\alpha(B)$ of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.*

Remarks .

1) As usual, the *diameter of a set B* is the number $\sup\{d(x, y) : x \in B, y \in B\}$ denoted by $\operatorname{diam}(B)$, with $\operatorname{diam}(\emptyset) = 0$. It is clear that $0 \leq \alpha(B) \leq \operatorname{diam}(B) < +\infty$ for each nonempty bounded subset B of X and that $\operatorname{diam}(B) = 0$ if and only if B is an empty set or consists of exactly one point.

2) The following properties are satisfied in any complete metric space and are a direct consequence of the definition:

(a) $\alpha(B) = 0$ if and only if B is precompact.
(b) $\alpha(\bar{B}) = \alpha(B)$ for all $B \in \mathcal{B}$.
(c) $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}, \quad \forall B_1 \in \mathcal{B}, \forall B_2 \in \mathcal{B}$.
(d) $B_1 \subset B_2$ implies $\alpha(B_1) \leq \alpha(B_2)$.
(e) $\alpha(B_1 \cap B_2) \leq \min\{\alpha(B_1), \alpha(B_2)\}, \quad \forall B_1 \in \mathcal{B}, \forall B_2 \in \mathcal{B}$.
(f) If B is a finite set, then $\alpha(B) = 0$.

The α-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem

THEOREM 1. *Kuratowski: Let (X, d) be a complete metric space and $\{B_n\}$ be a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(B_n) = 0$. Then the intersection B_∞ of all B_n is nonempty and compact.*

PROOF. Let $\{x_n\}$ be a sequence such that $x_n \in B_n$ for all $n \in \mathbb{N}$ and consider the decreasing sequence of sets $\{C_n\}$ given by $C_n = \{x_i : i \geq n\}$. Obviously $C_n \subset B_n$ and $\alpha(C_1) = \alpha(C_n) \leq \alpha(B_n)$ for every $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} \alpha(B_n) = 0$, it follows that $\alpha(C_1) = 0$ and so $\{x_n\}$ is a precompact set. Let $\overline{x}$ be the limit of a subsequence of $\{x_n\}$. Obviously $\overline{x} \in B_n$ for all $n \in \mathbb{N}$ and hence $B_\infty \neq \emptyset$.

Moreover, as $\alpha(B_\infty) \leq \alpha(B_n)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha(B_n) = 0$, we obtain $\alpha(B_\infty) = 0$ and so B_∞ is compact since it is a closed set.

□

Horvath (see [HO]) has proved the following generalization of the Kuratowski theorem.

THEOREM 2. *Let (X, d) be a complete metric space and $\{B_i\}_{i\in I}$ be a family of nonempty of closed and bounded subsets of X having the finite intersection property. If $\inf_{i\in I} \alpha(B_i) = 0$ then the intersection B_∞ of all B_i is nonempty and compact.*

PROOF. Given $n > 1$ there exists $i(n) \in I$ such that $\alpha(B_{i(n)}) < \frac{1}{n}$. Let $A_k = \cap_{n=1}^{k} B_{i(n)}$, then $\{A_k\}$ is is a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(A_k) = 0$, it follows by the Kuratowski theorem that $K = \cap_{k\geq 1} A_k$ is a nonempty compact subset of X.

Set $\hat{B}_i = B_i \cap K$ for each $i \in I$. It will be enough to prove that the family $\{\hat{B}_i\}_{i\in I}$ has the finite intersection property. In this situation, by compactness, $\cap_{i\in I}\hat{B}_i$ is nonempty, and since $\cap_{i\in I}\hat{B}_i \subset \cap_{i\in I} B_i$ the proof would be finished.

Let J be any non-empty finite subset of I and define

$$B_{J,1} = \cap_{i\in J} B_i, \quad B_{J,k} = \cap_{i\in J}(B_i \cap A_{k-1}) \ \text{ if } k \geq 2.$$

By the Kuratowski theorem $\cap_{k\geq 1} B_{J,k}$ is nonempty and compact. But $\cap_{k\geq 1} B_{J,k} = \cap_{i\in J} B_i \cap_{k\geq 1} A_k = \cap_{i\in J} B_i \cap K$. So the family $\{\hat{B}_i\}_{i\in I}$ has the finite intersection property and the proof is concluded.

□

COROLLARY 3. *Let (X, d) be a complete metric space and $f : X \to \mathbb{R}$ a lower semicontinuous function such that $\inf_{x\in X} \alpha\{y \in X : f(y) \leq f(x)\} = 0$. Then f is bounded from below and there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$.*

PROOF. Let $B_x = \{y \in X : f(y) \leq f(x)\}$, the family $\{B_x\}_{x\in X}$ is in the hypothesis of the previous theorem. If $x_0 \in \cap_{x\in X} B_x$, then $f(x_0) \leq f(x)$ for all $x \in X$.

□

COROLLARY 4. *Let (X, d) be a complete metric space and $g : X \to X$ a function such that:*

1) $x \to d(x, g(x))$ is lower semicontinuous.

2) $\inf_{x\in X} d(x, g(x)) = 0$.

3)$\inf_{x\in X} \alpha\{y \in X : d(y, g(y)) \leq d(x, g(x))\} = 0$.

Then there exists $x_0 \in X$ such that $g(x_0) = x_0$.

In the framework of the linear spaces can be deduced additional properties for the Kuratowski measure of noncompactness. We omit their well known proofs (see for instance [ADL2]).

PROPOSITION 5. *Let X be a Banach space, then we have:*

(a) $\alpha(tB) = |t|\alpha(B)$ for any real number t and $B \in \mathcal{B}$.

(b) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$, $\quad \forall B_1 \in \mathcal{B}$, $\forall B_2 \in \mathcal{B}$.

(c) $\alpha(x_0 + B) = \alpha(B)$ for any $x_0 \in X$ and $B \in \mathcal{B}$.

(d) $\alpha(B) = \alpha(\, co(B))$.

Associated to the notion of measure of noncompactness we can consider the class of condensing operators which generalize the notion of compact operator.

DEFINITION 2. *Let D be a nonempty subset of a metric spaces X. A mapping $T : D \subset X \to X$ is an α-condensing mapping if T is continuous and satisfies that $\alpha(T(A)) < \alpha(A)$ for every bounded subset A of D with $\alpha(A) > 0$.*

For the class of compact operators we have the following generalization of Schauder theorem, usually called *the theorem of Darbo and Sadovskiĭ* [D] and [Sa1].

Theorem 6. *Suppose M is a nonempty bounded closed and convex subset of a Banach space and suppose $T : M \to M$ is condensing. Then T has a fixed point.*

Proof. Fix $m \in M$ and denote by Σ the family of all closed and convex subsets K of M such that $m \in K$ and $T(K) \subset K$. Now set

$$B = \bigcap_{K \in \Sigma} K$$

and let

$$C = \overline{\mathrm{co}}(T(B) \cup \{m\}).$$

Obviously $\Sigma \neq \emptyset$, $m \in B$ and $T : B \to B$.

Moreover $B = C$. Indeed, since $m \in B$ and $T(B) \subset B$, it follows that $C \subset B$. This implies $T(C) \subset T(B) \subset C$ and so $C \in \Sigma$, and hence $B \subset C$. Using now the properties of α we obtain $\alpha(B) = \alpha(T(B))$. Since T is α-condensing, it follows that B is compact. Therefore T is a continuous mapping of the compact convex set B into itself. By Schauder theorem must have a fixed point.

□

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we have the definition of the Hausdorff measure.

Definition 3. *The Hausdorff measure of noncompactness $\chi(B)$ of the set B is the infimum of the numbers r such that B admits a finite covering by balls of radius smaller than r.*

Remarks:

3.) Recall that in a Banach space X , a set $S \subset X$ is called an r-net of B if $B \subset S + r\overline{B}(0,1) = \{s + rb : s \in S, b \in \overline{B}(0,1)\}$. So, the definition of the χ-measure in Banach spaces is equivalent to the following:

$$\chi(B) = \inf\{r > 0 : B \text{ has a finite } r-\text{net}\}.$$

4) The Hausdorff MNC enjoys the same properties as the Kuratowski MNC, appearing in Remarks 2, Proposition 5 and Theorems 1, 2 and 6 . However there is not a direct relationship between the Kuratowski and the Hausdorff MNCs as we will see in the following results.

Theorem 7. *Let $B(0,1)$ be the unit ball in a Banach space X. Then $\alpha(B(0,1)) = \chi(B(0,1)) = 0$ if X is finite dimensional, and $\alpha(B(0,1)) = 2$, $\chi(B(0,1)) = 1$ otherwise.*

Proof. If X is a finite dimensional Banach space, the result follows from the compactness of the unit ball.

It remains to consider the infinite dimensional case. We first prove the result for χ. Obviously $\chi(B(0,1)) \leq 1$. Suppose $\chi(B(0,1)) = r < 1$. Let us choose $\varepsilon > 0$ such that $r + \varepsilon < 1$. The there exist x_1, x_2,..., x_m in X such that

$$B(0,1) \subset \bigcup_{k=1}^{m} B\left(x_k, (r+\varepsilon)\right) = \bigcup_{k=1}^{m} (x_k + (r+\varepsilon)B(0,1)).$$

From the properties of the MNC χ, it follows that

$$r = \chi(B(0,1)) \le (r+\varepsilon)\chi(B(0,1)) = r(r+\varepsilon)$$

and this implies $r = 0$. So $\chi(B(0,1)) = 0$ and hence $B(0,1)$ is precompact. This contradicts the infinite dimensionality of the space X. Therefore, $\chi(B(0,1)) = 1$.

To prove the result for α we make use of the *Borsuk-Lyusternik-Shnirel'man theorem on antipodes* (see [Kr, page 100, Theorem 2.6]):

"If $S_n(0,1)$ is the unit sphere in an n-dimensional normed space and A_k ($k = 1,...,n$) is a cover of $S_n(0,1)$ by closed subsets of that space, then at least one of the sets A_k contains a pair of diametrically opposite points, that is, $\operatorname{diam}(A_k) \ge \operatorname{diam}(S_n(0,1))$".

Since $\operatorname{diam}(B(0,1)) = 2$, it is obvious that $\alpha(B(0,1)) \le 2$. Suppose $\alpha(B(0,1)) < 2$. Then we can find a finite number of closed subsets $\{B_1, B_2, ..., B_n\}$ of X with $\operatorname{diam}(B_k) < 2$ for all $k = 1,...,n$ such that $B(0,1) \subset \bigcup_{k=1}^n B_k$. Now, taking the section of $B(0,1)$ with an arbitrary n-dimensional subspace X_n and setting $A_k = B_k \cap X_n$, we obtain a contradiction with the theorem on antipodes.

□

Since α and χ are invariant under passage to the convex hull, we obtain the following corollary:

COROLLARY 8. *Let $S(0,1)$ be the unit sphere in a Banach space X. Then $\alpha(S(0,1)) = \chi(S(0,1)) = 0$ if X is finite dimensional, and $\alpha(S(0,1)) = 2$, $\chi(S(0,1)) = 1$ otherwise.*

THEOREM 9. *The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(B) \le \alpha(B) \le 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

PROOF. The inequalities are obvious from the definitions of α and χ. The sharpness of the second inequality follows from Theorem 7. The following example shows that the first inequality is also sharp. Let $B = \{e_k : k \ge 1\}$ the set of standard basis vectors in c_0. Since for all $i \ne j$, $\|e_i - e_j\| = 1$, we have $\alpha(B) = 1$. On the other hand, $\chi(B) = 1$ because the distance from any infinite subset of B to any element of c_0 cannot be smaller than 1.

□

Remark 5:

Even though in general α and χ are different MNCs, in some Banach spaces we can find a close relation between them. This is, for instance, the case of hyperconvex spaces (see chapter of Khamsi-Espinola).

2. ϕ-minimal sets and the separation measure of noncompactness

The notion of a ϕ-minimal set, where ϕ is the α or χ MNCs, was introduced in [Do1] in order to study the relationships between condensing mappings for Kuratowski and Hausdorff's measures of noncompactness. The notion has became very useful to study some properties of the Separation measure of noncompactness and to obtain simpler forms for some moduli of noncompact convexity and to simplify several proofs.

Definition 4. *Let X be a metric space and $\mathcal{B}$ the family of all bounded subsets of X. An infinite subset $A \in \mathcal{B}$ is said to be minimal for the measure ϕ (or, in short, ϕ-minimal) if $\phi(A) = \phi(B)$ for every infinite subset B of A.*

Examples :
(1) Every infinite precompact set is obviously ϕ-minimal.
(2) In particular, every Cauchy sequence with infinite range is ϕ-minimal.
(3) Every infinite subset of a ϕ-minimal set is a ϕ-minimal set.
(4) The standard bases in ℓ^p or c_0 are minimal sets.
(5) A nonprecompact convex subset of a Banach space fails to be ϕ-minimal.

We are now going to prove the existence of ϕ-minimal sets in bounded sets.

Theorem 10. *Let X be a bounded metric space. Then:*
(a) There is a subset A of X such that A is ϕ-minimal.
(b) If X is not a precompact set, A can be chosen such that $\phi(A) > 0$.

Proof. Let $A_0 = X$; recursively let

$$\phi_{n+1} = \inf\{\phi(A) : A \subset A_n,\ A \text{ infinite}\}$$

and let A_{n+1} be chosen to be an infinite subset of A_n with

$$\phi(A_{n+1}) < \phi_{n+1} + \frac{1}{n+1}.$$

Since A_n is an infinite set, for every n we can choose $a_n \in A_n$ such that $a_n \neq a_k$ for $k = 1, 2, ..., n-1$. Let A be the infinite set $\{a_n : n \in \mathbb{N}\}$. Then $A \backslash A_n$ is finite for each $n \in \mathbb{N}$. Let us see that A is ϕ-minimal. Indeed, let A' be an infinite subset of A. Since $A' \backslash A_{n-1}$ is a finite set for each $n > 1$, we have

$$\phi(A) \leq \phi(A_n) < \phi_n + \frac{1}{n} \leq \phi(A' \cap A_{n-1}) + \frac{1}{n} = \phi(A') + \frac{1}{n}.$$

Hence $\phi(A') = \phi(A)$ and A is ϕ-minimal. This argument concludes part (a).

We now assume that $\phi(A) = 0$ for every ϕ-minimal subset A of X. Let $\{x_n\}$ be a sequence in X and we assume that the set $\{x_n : n \in \mathbb{N}\}$ is infinite. Then there exists an ϕ-minimal subsequence $\{y_n\}$ of $\{x_n\}$. Since $\phi(\{y_n : n \in \mathbb{N}\}) = 0$, $\{y_n : n \in \mathbb{N}\}$ is precompact and so $\{y_n\}$ has a Cauchy subsequence. Thus every sequence in X has a Cauchy subsequence. Hence X is a precompact set and the proof is complete.

$\square$

Theorem 11. *Let X be a separable metric space and B a bounded subset of X. Then there exists a χ-minimal subset A of B such that $\chi(A) = \chi(B)$.*

Proof.

Whitout loss of generality, assume that $\chi(B) > 0$. Let $\{p_k\}$ be a dense sequence in X and let $\{B_m\}$ be the sequence formed by all balls centered at each p_k with radius a rational positive number less than $\chi(B)$. Then for every positive integer n there are infinitely many points which are not in $M_n = \cup_{m=1}^n B_m$.

Choose $x_n \in B \setminus M_n$ and denote by A the set $\{x_n : n \in \mathbb{N}\}$. For every n there exists $k(n) > n$ such that $x_j \in M_{k(n)}$ for $j = 1, ..., n$. Hence $x_{k(n)} \neq x_j$, $j = 1, ..., n$. Therefore A is an infinite set.

Let C be an infinite subset of A. For each $n \in \mathbb{N}$, C is not contained in M_n because $\cap M_n$ is a finite set. Now it is easy to prove that $\chi(C) \geq \chi(B) \geq \chi(A)$. Thus A is χ-minimal and $\chi(B) = \chi(A)$

$\square$

Remark 6. This result hold for a wide class of spaces. We recall that a Banach space X is said to be *weakly compactly generated* if it contains a weakly compact set which is linearly dense in X. Obviously, reflexive and separable Banach spaces satisfy this condition. S. Prus (see [ADL2]) has proved that the previous theorem is also true for weakly compactly generated Banach spaces. However, this result does not hold for ℓ_∞ (se [Do1]).

The situation for the Kuratowski MNC is different as we are going to show. We need some technical results

LEMMA 12. *Let A be an α-minimal subset of a metric space (X, d). Then, for every positive number ε, there exists an infinite subset B of A such that*

$$\alpha(A) - \varepsilon < d(x, y) < \alpha(A) + \varepsilon$$

for every $x \in B$, $y \in B$, $x \neq y$.

PROOF. Without loss of generality we can assume that card(A)= χ_0 and $d(x, y) < \alpha(A)+\varepsilon$ for every $x, y \in A$. By Ramsey's theorem (see, for example, [Ba, pag. 392]) there exists an infinite subset B of A such that $d(x, y) \leq \alpha(A)-\varepsilon$ for every $x, y \in B$ or $d(x, y) \geq \alpha(A) - \varepsilon$ for every $x, y \in B$. Since the first possibility contradicts the α-minimality of A and the proof is concluded.

$\square$

Remark 7. For a proof of the above lemma without using Ramsey's theorem see [ADL2, Lemma 3.1.3].

The following theorem which can be deduced from lemma 12 is known (see [DU, pag 235])

THEOREM 13. *Let $\{x_n\}$ be a bounded sequence in a metric space (X, d). Then there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\lim_{n,m;n\neq m} d(y_n, y_m)$ exists.*

Let $\{x_n\}$ be a bounded sequence in a Hilbert space H. Since H is reflexive, there is a subsequence $\{y_n\}$ of $\{x_n\}$ and a vector $v \in H$ such that $\{y_n\}$ is weakly convergent to v. Taking a subsequence if necessary, we can also assume that $\lim_{n\to\infty} \|y_n - v\|$ exists.

Consider the mapping $\Phi : H \to \mathbb{R} : \Phi(z) = \limsup_{n\to\infty} \|y_n - z\|$. This map takes its unique absolute minimum in v. This result is a consequence of the following lemma due to Opial [O1]:

LEMMA 14. *If in a Hilbert space H the sequence $\{x_n\}$ is weakly convergent to x_0, then for any $x \neq x_0$ we have*

$$\limsup_{n\to\infty} \|x_n - x\| > \limsup_{n\to\infty} \|x_n - x_0\|. \tag{1}$$

PROOF. Since every weakly convergent sequence is necessarily bounded, both limits in (1) are finite. Thus, to prove this inequality, it suffices to observe that in the equality

$$\|x_n - x\|^2 = \|x_n - x_0\|^2 + \|x_0 - x\|^2 + 2(x_n - x_0)\cdot(x_0 - x)$$

the last term tends to zero as n tends to infinity.

□

Now it is easy to deduce that $\chi(\{y_n : n \in \mathbb{N}\}) = \Phi(v)$. Indeed, since $\Phi(v) = \lim_{n\to\infty} \|y_n - v\|$ it follows that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $y_n \in B(v, \Phi(v) + \varepsilon)$ for all $n \geq n_0$, and hence $\chi(\{y_n : n \in \mathbb{N}\}) \leq \Phi(v)$.

Conversely, let us suppose that $\{y_n : n \in \mathbb{N}\}$ can be covered by finitely many balls with radius $r < \Phi(v)$. Then there is a ball $B(u, r)$ containing infinitely many elements of this sequence. We still denote the subsequence contained in this ball by $\{y_n : n \in \mathbb{N}\}$. This sequence is weakly convergent to v, $\Phi(v)$ is still given by $\lim_{n\to\infty} \|y_n - v\|$ and the function $\Phi' : H \to \mathbb{R} : \Phi'(z) = \limsup_{n\to\infty} \|y_n - z\|$ takes its unique absolute minimum in v. Therefore we obtain $\|y_n - u\| \leq r < \Phi(v)$ for all $n \in \mathbb{N}$ and thus $\Phi'(u) = \limsup_{n\to\infty} \|y_n - u\| \leq r < \Phi(v) = \Phi'(v)$ contradicting the fact that Φ' has an absolute minimum at v.

LEMMA 15. *Let $\{x_n\}$ be an α-minimal sequence in a Hilbert space H. Assume that $x_n \neq x_m$ if $n \neq m$. If $\{y_n\}$, v and Φ are constructed as above, then*

$$\alpha(\{x_n : n \in \mathbb{N}\}) = \alpha(\{y_n : n \in \mathbb{N}\}) = \Phi(v)\sqrt{2} = \chi(\{y_n : n \in \mathbb{N}\})\sqrt{2}.$$

PROOF. We only have to prove the equality $\alpha(\{y_n : n \in \mathbb{N}\}) = \Phi(v)\sqrt{2}$.

Let ε be an arbitrary positive number and define $\eta = \alpha(\{y_n : n \in \mathbb{N}\})$. By Lemma 12 there is a subsequence $\{z_n\}$ of $\{y_n\}$ such that

$$\eta - \varepsilon \leq \|z_n - z_m\| \leq \eta + \varepsilon, \ \forall n \neq m. \tag{2}$$

Since $\lim_{n\to\infty} \|y_n - v\| = \Phi(v)$, we can choose a positive integer k such that

$$|\|z_n - v\| - \Phi(v)| < \varepsilon, \ n \geq k. \tag{3}$$

Fixing $k \in \mathbb{N}$, let $n > k$ be large enough so that $|(z_n - v)\cdot(z_k - v)| < \varepsilon$. From the identity

$$\|z_k - z_n\|^2 = \|z_k - v\|^2 + \|z_n - v\|^2 - (z_k - v)\cdot(z_n - v) - (z_n - v)\cdot(z_k - v)$$

we obtain by (2) and (3)

$$2(\Phi(v) + \varepsilon)^2 + 2\varepsilon \geq (\eta - \varepsilon)^2 \ \text{ and } \ (\eta + \varepsilon)^2 \geq 2(\Phi(v) - \varepsilon)^2 - 2\varepsilon.$$

Hence, letting $\varepsilon \to 0$ we obtain $\eta = \sqrt{2}\Phi(v)$.

□

THEOREM 16. *Let H be an infinite dimensional Hilbert space and A an α-minimal subset of the unit sphere $S(0,1)$. Then*

$$\alpha(A) \leq \sqrt{2} < 2 = \alpha(S(0,1)).$$

PROOF. Let $\{x_n\}$ be a sequence in A with $x_n \neq x_m$ for all $n \neq m$. By Lemma 15 we have $\frac{\alpha(A)}{\sqrt{2}} = \inf\{\Phi(z) : z \in H\}$, where $\Phi(z) = \limsup_{n\to\infty} \|y_n - z\|$ for a subsequence $\{y_n\}$ of $\{x_n\}$. Then $\frac{\alpha(A)}{\sqrt{2}} \leq \Phi(0) = \limsup_{n\to\infty} \|y_n\| = 1$. On the other hand we know from Corollary 8 that $\alpha(S(0,1)) = 2$.

□

The notion of minimal set is strongly related to the Separation measure of noncompactness which was first introduced by Istrastescu [I1]. The set B is said to be *r-separated* if $d(x,y) \geq r$ for all $x, y \in B$, $x \neq y$. The set B will be call an *r-separation.*

DEFINITION 5. *The measure of noncompactness $\beta(B)$ of the subset B of the metric space X is the infimum of the numbers $r > 0$ for which B does not have an infinite r-separation or, equivalently, the supremun of those $r > 0$ for which B has an infinite r-separation.*

Remarks:
8.) It is easy to prove that

$$\beta(B) = \sup\{\alpha(A) : A \subset B \text{ and A is } \alpha\text{-minimal } \}.$$

9.) The properties of the separation measure of noncompactness and related fixed point theorems has been studied by many authors(see ADL2). The permanence under passage to the convex hull was proved independently by J. Arias [A] and ...

3. Moduli of noncompact convexity.

A generalization of the moduli of k-uniform convexity, (see chapter XX (Prus)), considering infinite dimensional subspaces, was given by Goebel and Sękowski in 1984 [GS]. The corresponding concept to k-uniformly convex space is, in this case, that of nearly uniformly convex space. Goebel and Sękowski use the Kuratowski MNCs as a measure of the volume of the sets. The use of the measure of noncompactness is relevant since it implies a strong connection between topological methods and metric methods.

We are going to start this section with the definition of the moduli of noncompact convexity. Throghout this section ϕ will denote the α, χ or β measure of nocompactness.

DEFINITION 6. *Let X be a Banach space. We define the modulus of noncompact convexity associated to ϕ in the following way*

$$\begin{aligned}\Delta_{X,\phi}(\varepsilon) &= \inf\{1 - \ d(0,A) : A \subset \overline{B}(0,1) \textit{ is convex }, \ \phi(A) > \varepsilon\} \\ &= \inf\{1 - \ d(0, \ co(A)) : A \subset \overline{B}(0,1), \phi(A) > \varepsilon\}.\end{aligned}$$

We define the NUC-characteristic of X associated to the measure of noncompactness ϕ to be

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The function $\Delta_{X,\alpha}(\varepsilon)$ has been considered by Goebel and Sękowski [GS], $\Delta_{X,\chi}(\varepsilon)$ by Banaś [Ba1] and $\Delta_{X,\beta}(\varepsilon)$ by Domínguez and López [DL2].

Remarks:

10.) A Banach space is NUC if and only if $\varepsilon_\phi(X) = 0$.

11.) The following relationships among the different moduli are easy to obtain

$$\delta_X(\varepsilon) \leq \Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon)$$

and consequently

$$\varepsilon_0(X) \geq \varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X).$$

We are going to introduce some equivalent definitions for the moduli $\Delta_{X,\chi}(\varepsilon)$ and $\Delta_{X,\beta}(\varepsilon)$ in the case of reflexive spaces. The following technical lemma can be easily proved.

Lemma 17. *Let X be a Banach space and $\{x_n\}$ a sequence in X weakly convergent to w. Let $A_n = \overline{co}(\{x_k\}_{k\geq n})$. Then*

$$\bigcap_{n=1}^{\infty} A_n = \{w\}.$$

Theorem 18. *Let X be a reflexive Banach. Then*

$$\Delta_{X,\beta}(\varepsilon) = \inf\left\{1 - \|w\| : \{x_n\} \subset \overline{B}(0,1), x_n \rightharpoonup w \ \textit{and} \ \ \textit{Sep}(\{x_n\}) > \varepsilon\right\}$$

where

$$\textit{Sep}(\{x_n\}) = \inf\{\varepsilon : \|x_n - x_m\| \geq \varepsilon : n \neq m\}$$

and

$$\Delta_{X,\chi}(\varepsilon) = \inf\left\{1 - \|w\| : \{x_n\} \subset \overline{B}(0,1), x_n \rightharpoonup w \ \textit{and} \ \chi(\{x_n\}) > \varepsilon\right\}.$$

Proof. Both equalities have a similar proof. So we are only going to prove the first one.

First we note that if $A \subset \overline{B}(0,1)$ with $\beta(A) > \varepsilon$, there exists a sequence $\{x_n\}$ such that $\mathrm{Sep}(\{x_n\}) > \varepsilon$, $\{x_n\} \subset A$ and $\{x_n\} \rightharpoonup w$. Thus

$$\Delta_{X,\beta}(\varepsilon) = \inf\{1 - \ \mathrm{d}(\overline{\mathrm{co}}(\{x_n\}), 0) : x_n \rightharpoonup w, \{x_n\} \subset \overline{B}(0,1) \text{ and } \ \mathrm{Sep}(\{x_n\}) > \varepsilon\}$$

Now using the previous lemma it is easy to deduce the first inequality:

$$\Delta_{X,\beta}(\varepsilon) \geq \inf\left\{1 - \|w\| : x_n \rightharpoonup w, \{x_n\} \subset \overline{B}(0,1) \text{ and } \ \mathrm{Sep}(\{x_n\}) > \varepsilon\right\}.$$

Conversely, let $x_n \rightharpoonup w$ with $\{x_n\} \subset \overline{B}(0,1)$ and $\operatorname{Sep}(\{x_n\}) > \varepsilon$. As

$$1 - \Delta_{X,\beta}(\varepsilon) = \sup\left\{ \, \mathrm{d}(0, \overline{\mathrm{co}}(\{y_n\}) : \{y_n\} \subset \overline{B}(0,1) \quad \operatorname{Sep}(\{y_n\}) > \varepsilon \right\},$$

if we take $\{y_n\} = \{x_n\}_{n \geq k}$, we have

$$1 - \Delta_{X,\beta}(\varepsilon) \geq \ \mathrm{d}(0, \overline{\mathrm{co}}\,(\{x_n\}_{n\geq k})).$$

Let $\eta > 0$. For all k there exists

$$z_k \in \overline{\mathrm{co}}(\{x_n\}_{n\geq k}) = A_k$$

such that

$$\|z_k\| \leq 1 - \Delta_{X,\beta}(\varepsilon) + \eta.$$

Let z_{n_k} be a weakly convergent subsequence of $\{z_n\}$, and $B_n = \overline{\mathrm{co}}\{x_{n_k}\}_{k\geq n}$. Then $B_n \subset A_n$. So if $z_{n_k} \rightharpoonup z$ we obtain

$$z \in \bigcap_{n=1}^{\infty} B_n \subset \bigcap_{n=1}^{\infty} A_n = \{w\}.$$

Thus $z_n \rightharpoonup w$ and so

$$\|w\| \leq \liminf_{n\to\infty} \|z_n\| \leq 1 - \Delta_{X,\beta}(\varepsilon) + \eta.$$

Since the last inequality is true for every η, we obtain

$$\Delta_{X,\beta}(\varepsilon) \leq 1 - \|w\|$$

and again this inequality is true for every $x_n \rightharpoonup w$ satisfying $\{x_n\} \subset \overline{B}(0,1)$ and $\operatorname{Sep}(\{x_n\}) > \varepsilon$. So

$$\Delta_{X,\beta}(\varepsilon) \leq \inf\{1 - \|w\| : x_n \rightharpoonup w, (x_n) \subset \overline{B}(0,1) \quad \operatorname{Sep}(\{x_n\}) > \varepsilon\}$$

and the proof is now complete.

$\square$

Remarks:
12.) The function

$$P_X(\varepsilon) = \inf\left\{1 - \|w\| : x_n \rightharpoonup w, \{x_n\} \subset \overline{B}(0,1) \text{ and } \operatorname{Sep}(\{x_n\}) > \varepsilon\right\}$$

has been considered by Partington [Pa] as a modulus for the UKK property. Theorem 17 shows that for reflexive spaces Partington's modulus is identical to the modulus of noncompact convexity associated to β.

13.) Let X be a Banach space and τ a topology on X. Bearing in mind Theorem 17 it is natural to consider the following moduli (for more details see Japon)

$$\Delta_{X,\beta,\tau}(\varepsilon) = \inf\left\{1 - \|w\| : \tau - \lim\{x_n\} = w, \{x_n\} \subset \overline{B}(0,1), \text{ and } \operatorname{Sep}(\{x_n\}) > \varepsilon\right\}$$

and

$$\Delta_{X,\chi,\tau}(\varepsilon) = \inf\left\{1 - \|w\| : \tau - \lim\{x_n\} = w, \{x_n\} \subset \overline{B}(0,1), \text{ and } \chi(\{x_n\}) > \varepsilon\right\},$$

where $\tau - \lim\{x_n\}$ denotes the limit of a sequence $\{x_n\}$ with respect to the topology τ

Notice that a Banach X has the UKK(τ) property (see definition 8 in Chapter Prus) if and only if $\Delta_{X,\beta,\tau}(\varepsilon) > 0$ for all $\varepsilon > 0$.

Example 6. Let (Ω, Σ, μ) be a positive σ-finite measure space, $X = (L_p(\mu), \|.\|_p)$ and τ the convergence locally in measure toplogy (clm) (see chapter Prus for more details). We are going to prove that:

$$\Delta_{X,\beta,clm}(\varepsilon) = 1 - (1 - (\varepsilon)^p)^{\frac{1}{p}}.$$

for $1 \leq p < +\infty$.

We use the following result which can be derived from [BL]:

If $\{f_n\}$ is a clm-null sequence in $L_p(\mu)$, $1 \leq p < +\infty$, and f is a function in $L_p(\mu)$, then

$$\limsup_n \|f_n - f\|_p^p = \|f\|_p^p + \limsup_n \|f_n\|_p^p. \tag{4}$$

Let $\{f_n\}$ be a sequence which is clm-convergent to $f \in L_p(\mu)$. Suppose that $\|f_n\|_p \leq 1$ and $\text{Sep}(\{f_n\}) > \varepsilon$. Take a subsequence $\{g_n\} \subset \{f_n\}$ such that $\lim_n \|g_n - f\|_p$ and $\lim_n \|g_n\|_p$ exist. If we denote $h_n = g_n - f$ and apply (4) to the sequence $\{h_n\}$ and $-f$, we obtain:

$$\begin{aligned}\|f\|_p^p &= \lim_n \|h_n + f\|_p^p - \lim_n \|h_n\|_p^p = \lim_n \|g_n\|_p^p - \lim_n \|g_n - f\|_p^p \\ &\leq 1 - \lim_n \|g_n - f\|_p^p.\end{aligned}$$

Denote $l = \lim_n \|g_n - f\|_p$ and let $m \in \mathbb{N}$. We have

$$\varepsilon^p \leq \limsup_n \|g_n - g_m\|_p^p = \limsup_n \|g_n - f - (g_m - f)\|_p^p.$$

Applying again (4) we obtain

$$\varepsilon^p \leq \|g_m - f\|_p^p + \lim_n \|g_n - f\|_p^p.$$

Taking the limit when m tends to infinity, we deduce $l \geq \varepsilon/2^{\frac{1}{p}}$. Hence $\|f\|_p^p \leq (1 - \varepsilon^p/2)^{\frac{1}{p}}$ and

$$\Delta_{X,\beta,clm}(\varepsilon) \geq 1 - (1 - (\varepsilon)^p)^{\frac{1}{p}}.$$

Let us now study the converse inequality.

Suppose $\Omega = \cup_{n=1}^{\infty} \Omega_n$ with $\mu(\Omega_n) < \infty$. Given $\eta > \varepsilon$ consider the function $f = a\chi_{\Omega_1}$ where $a^p = (1 - \eta^P/2)/\mu(\Omega_1)$ and the sequence $f_n = f + b_n \chi_{\Omega_n}$ where $b_n^p = \eta^p/(2\mu(\Omega_n))$ for $n \geq 2$. It is clear that $clm - \lim_n f_n = f$, $\|f\|_p = 1$, $\|f_n - f_m\|_p = \eta$ and $\|f\|_p = (1 - \eta^P/2)^{\frac{1}{p}}$. Since η is any real number greater that ε, we finally obtain the desired equality.

Using similar arguments (see [J]) we can also compute the modulus for the case of the measure χ

$$\Delta_{X,\chi,clm}(\varepsilon) = 1 - (1 - (\varepsilon)^p)^{\frac{1}{p}}.$$

Remarks:

14.) Suppose that $\Omega = \mathbb{N}$ and μ is the counting measure. Then we obtain the space ℓ^p and *clm* convergence of a sequence is equivalent to weak convergence for bounded sequences if $p > 1$. So

$$\Delta_{\ell^p,\beta}(\varepsilon) = 1 - (1 - (\varepsilon)^p)^{\frac{1}{p}}.$$

and

$$\Delta_{\ell^p,\chi}(\varepsilon) = 1 - (1 - (\varepsilon)^p)^{\frac{1}{p}},$$

for $1 \leq p < +\infty$

If $p = 1$ clm convergence is equivalent to weak* convergence for bounded sequences if ℓ^1 is seen as the dual space of c_0.

15.) Goebel and Sękowski [GS] computed the value of $\Delta_{\ell^p,\alpha}$ obtaining

$$\Delta_{\ell^p,\alpha}(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}$$

which, for $p \geq 2$, coincides with the value of the Clarkson modulus of convexity (see chapter Prus).

4. Fixed point theorems derived from normal structure

In this section we are going to study fixed point results for nonexpansive mappings. Recall that if (C, d) is a metric space, a mapping $T : C \to C$ is said to be *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for every $x, y \in C$. The framework where we are going to work is the following: Let X be a Banach space and τ an arbitrary topology considered on X. We shall say that X has the *fixed point property with respect to* τ (τ-FPP), if every nonexpansive mapping defined from a τ-sequentially compact bounded convex subset of X into itself has a fixed point. Notice that according to the Eberlein-Smulian Theorem, which proves the equivalence between weak compactness and weak sequential compactness, the previous definition is equivalent to the usual definition of the *weak fixed point property* whenever τ is the weak topology. Our main goal in this section is to obtain fixed point results derived from normal structure. We will finish using the moduli of noncompact convexity to deduce fixed point results for nonexpansive mappings.

We shall say that a Banach space X has *normal structure with respect to a topology* τ (τ-NS), if every τ-sequentially compact bounded convex set $C \subset X$ with $\operatorname{diam}(C) > 0$ contains a nondiametral point, i.e., there is $x_0 \in C$ such that $\sup\{\|x - x_0\| : x \in C\} < \operatorname{diam}(C)$. This notion has been defined previously in Chapter (Prus).

Before starting the main theorem of this section we need to prove the following Lemma which can be found essentially in [GW], although the proof that we give is from [Ki3]. Recall that the Chebyschev radius of a set C is defined as

$$r(C) = \inf_{x \in C}\{\sup \|x - y\| : y \in C\}$$

LEMMA 19. *Let X be a Banach space and C a bounded subset. Let $\mathcal{S}$ be a class of subsets of C which is stable under arbitrary intersections and it contains $B \cap C$ for every closed ball B of X. If $T : C \to C$ is a nonexpansive mapping, for every $\epsilon > 0$ there exists $C(\epsilon) \in \mathcal{S}$ such that $T : C(\epsilon) \to C(\epsilon)$ and $diam(C(\epsilon)) \leq r(C) + \epsilon$.*

PROOF. If $\operatorname{diam}(C) = 0$, take $C(\epsilon) = C$. Otherwise, let $\rho = r(C) + \epsilon$. We are going to construct $C(\epsilon)$.

Firstly, by definition of ρ, the set $M = \{z \in C : C \subset B(z, \rho)\}$ is a nonempty set. Let

$$\mathcal{F} = \{D \in \mathcal{S} : M \subset D, T : D \to D\}.$$

Notice that $\mathcal{F} \neq \emptyset$ since $C \in \mathcal{F}$. Define

$$L = \bigcap_{D \in \mathcal{F}} D$$

Then $L \in \mathcal{S}$, $M \subset L$ and $T : L \to L$. Let $A = M \cup T(L)$. Let us prove that

$$L = \bigcap \{D \in \mathcal{S} : A \subset D\}.$$

Since $A \subset L$ it is clear that $\cap\{D \in \mathcal{S} : A \subset D\} \subset L$. This implies that $T(\cap\{D \in \mathcal{S} : A \subset D\}) \subset T(L) \subset A \subset \cap\{D \in \mathcal{S} : A \subset D\}$. Also $M \subset A \subset \cap\{D \in \mathcal{S} : A \subset D\}$. Thus the last set belongs to $\mathcal{F}$ and contains the set L.

Finally we define

$$C(\epsilon) = \{x \in L : L \subset B(x, \rho)\} = \left\{ \bigcap_{u \in L} B(u, \rho) \right\} \cap L$$

Notice that $C(\epsilon) \neq \emptyset$ since $M \subset C(\epsilon)$. On the other hand, $C(\epsilon) \in \mathcal{S}$ because it is an arbitrary intersection of sets in $\mathcal{S}$. It is also clear that $\operatorname{diam}(C(\epsilon)) \leq \rho$. We need to prove that $C(\epsilon)$ is T-invariant.

Let $x \in C(\epsilon)$. Obviously $T(x) \in L$ so we have to prove that $L \subset B(Tx, \rho)$. Firstly, we shall see that $A \subset B(Tx, \rho)$. If $u \in M$ we have $u \in B(Tx, \rho)$ because $C \subset B(u, \rho)$. Otherwise, if $u \in T(L)$ there exists $y \in L$ such that $u = T(y)$. Hence $d(u, Tx) \leq d(y, x) \leq \rho$ because $x \in C(\epsilon)$. Consequently, $A \subset B(Tx, \rho) \cap L$ and by the equivalent definition of L we finally deduce that $L \subset B(Tx, L)$ and $C(\epsilon)$ is T-invariant.

□

Recall that if X is a Banach space and τ is a topology on X, it is said that the norm is τ-sequentially lower semicontinuous (τ-slsc) if $\|x\| \leq \liminf_n \|x_n\|$ whenever $\tau - \lim_n x_n = x$.

THEOREM 20. *Let X be a Banach space and τ a topology on X such that the norm $\|\cdot\|$ is a τ-slsc function. If X has τ-NS, then X satisfies the τ-FPP.*

PROOF. Let C be a nonempty τ-sequentially compact bounded convex subset of X and let $T : C \to C$ be a nonexpansive mapping.

Denote $\mathcal{A}$ the following family of subsets of C:

$$\mathcal{A} = \{K \subset C : \, K \neq \emptyset \ \tau\text{-sequentially closed, convex and } T : K \to K\}.$$

Notice that $\mathcal{A} \neq \emptyset$ since $C \in \mathcal{A}$. For every $K \in \mathcal{A}$ we define

$$\delta(K) = \inf\{\mathrm{diam}(F) : F \in \mathcal{A}, F \subset K\}$$

Set $K_1 = C$, and with $K_1, ..., K_n$ given, select $K_{n+1} \in \mathcal{A}$ such that $K_{n+1} \subset K_n$ and

$$\mathrm{diam}(K_{n+1}) \leq \delta(K_n) + \frac{1}{n}$$

for every $n \in \mathbb{N}$.

Let $K_0 = \cap_{n=1}^{\infty} K_n$. Firstly, we are going to check that $K_0 \neq \emptyset$: Take $x_n \in K_n$ for every $n \in \mathbb{N}$ and consider the sequence $\{x_n\}$. Since C is a τ-sequentially compact set, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ which is τ-convergent to a point x. Since the sets K_n are τ-sequentially closed and $K_{n+1} \subset K_n$, we know that $x \in K_n$ for every $n \in \mathbb{N}$ and so, $K_0 \neq \emptyset$. On the other hand, K_0 is convex, τ-sequentially compact and T-invariant.

Define

$$\mathcal{S} = \{D \subset K_0 : \ D \text{ is convex and } \tau\text{-sequentially closed}\}$$

Clearly, $\mathcal{S} \neq \emptyset$ since $K_0 \in \mathcal{S}$, $\mathcal{S}$ is stable under arbitrary intersections and $B \cap C \in \mathcal{S}$ for every closed ball in X due to the τ-sequential lower semicontinuity of the norm.

The above lemma implies that for each $\epsilon > 0$ there exists $K_0(\epsilon) \in \mathcal{S}$ T-invariant and such that

$$\mathrm{diam}(K_0) - \frac{1}{n} \leq \mathrm{diam}(K_{n+1}) - \frac{1}{n} \leq \delta(K_n) \leq \mathrm{diam}(K_0(\epsilon)) \leq r(K_0) + \epsilon$$

Letting $n \to \infty$

$$\mathrm{diam}(K_0) \leq r(K_0) + \epsilon$$

Since this is true for each $\epsilon > 0$ we deduce that $\mathrm{diam}(K_0) = r(K_0)$ which implies that $K_0 = \{x\}$ since X has τ-NS. Therefore $T(x) = x$.

$\square$

Remark 16 Notice that Theorem 20 generalizes Kirk's Theorem [Ki1] for general topologies because the norm is always a w-slsc function.

On the other hand, the existence of fixed points for nonexpansive mappings defined on domains which are compact with respect to certain topology, has been studied in several papers [Be], [LT], [Le], [Kh3], [KT]... All of them require, as additional assumptions, the τ-sequential compactness of the domain. From Theorem 20 we check that the τ-compactness is not necessary to assure fixed point result in the above hypothesis

Next we are going to define a geometric property in a Banach space which implies normal structure with respect to a topology τ in very general conditions.

DEFINITION 7. *Let X be a Banach space and τ a topology on X. We shall say that X has the τ-GGLD property if:*

$$\lim_n \|x_n\| < \lim_{n,m;n\neq m} \|x_n - x_m\|$$

for every bounded τ-null sequence $\{x_n\}$ such that both limits exist and $\lim_n \|x_n\| \neq 0$.

THEOREM 21. *Let X be a separable Banach space. If τ is a linear topology and X has the τ-GGLD property then X has τ-NS. In case $\tau \equiv w$ the separability assumption is not necessary.*

PROOF. If X has not τ-NS there is a bounded convex τ-sequentially compact subset B which satisfies $r(B) < \operatorname{diam}(B)$ where $r(B)$ is the Chebyshev radius of B. As in Corollary 52 in Chapter (Prus) we can obtain a sequence $\{x_n\} \subset B$ which converges to x with respect to τ and

$$\lim_n \|x_n - x\| = \lim_n \|x_n - x_m\|$$

for every $m \in \mathbb{N}$. Applying Theorem 13 there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that $\lim_{n,m;n\neq m} \|y_n - y_m\|$ exists. Finally, if we set $z_n = y_n - x$ it is clear that the sequence $\{z_n\}$ is bounded, τ-null convergent since τ is a linear topology and also

$$\lim_n \|z_n\| = \lim_{n,m;n\neq m} \|z_n - z_m\|$$

which contradicts the τ-GGLD property.

Notice that if τ is the weak topology the separability is not necessary in the same way as in Corollary 52 in Chapter (Prus).

□

Recall that if X is a Banach space and τ a topology on X the following coefficient was defined in Chapter (Prus)

$$\tau CS(X) = \inf \left\{ \frac{\lim_{n,m;n\neq m} \|x_n - x_m\|}{\lim_n \|x_n\|} \right\}$$

where the infimum is taken over all norm bounded sequences which converge to 0 in τ, both limits exist and $\lim \|x_n\| \neq 0$. Property τ-GGLD is related to the coefficient $\tau CS(X)$ in the following way: X has the τ-GGLD property if $\tau CS(X) > 1$. Consequently, we can state the following result which relates the τ-FPP and the coefficient $\tau CS(X)$.

COROLLARY 22. *Let X be a separable Banach space and τ a linear topology on X such that the norm is a τ-slsc function. If $\tau CS(X) > 1$ then X has the τ-FPP. If $\tau \equiv w$ the separability assumption is not necessary.*

Remark 17

If $X = (L_p(\mu), \|.\|_p)$) and τ is the clm-topology then $\tau CS(X) = 2^{1/p}$ for $1 \leq p < \infty$ (see chapter Prus). So as a consequence of Corollary 22 we can deduce that $L_1(\mu)$ has the clm-FPP although fails the $weak$-FPP (see [Al]). This result had been proved previously by Lennard ([Le]).

To finish this section, we shall see how the coefficient $\tau CS(X)$ is related to the moduli of noncompact convexity defined in Remark 13 in Section 3. As a consequence the moduli of noncompact convexity are also useful to deduce fixed point results for nonexpansive mappings.

LEMMA 23. *Let X be a Banach space and τ a linear topology on X. Then:*

$$\tau CS(X) \geq \lim_{\varepsilon \to 1^-} \frac{1}{1 - \Delta_{X,\beta,\tau}(\varepsilon)} = \frac{1}{1 - \Delta_{X,\beta,\tau}(1^-)}$$

PROOF. Let $\epsilon < 1$ be an arbitrary positive number and $\{x_n\}$ a sequence τ-convergent to zero, such that $\lim_n \|x_n\|$ exists and $\lim_{n,m;n\neq m}\|x_n - x_m\| = (\epsilon+1)/2$. If we take $\epsilon < \eta < (\epsilon+1)/2$, we can find $k_0 \in \mathbb{N}$ such that $\eta \leq \|x_n - x_m\| \leq 1$ if $m, n \geq k_0$.

Consider $m \geq k_0$ and the sequence $\{y_n\} = \{x_n - x_m\}_{n\geq k_0}$ which is τ-convergent to the point $-x_m$. In addition, the sequence $\{y_n\}$ is contained in the closed unit ball of X and $\operatorname{sep}(\{y_n\}) \geq \eta > \epsilon$. From the definition of the modulus $\Delta_{X,\beta,\tau}(\cdot)$, we deduce that $\|x_m\| \leq 1 - \Delta_{X,\beta,\tau}(\epsilon)$. Letting $m \to +\infty$ we obtain:

$$\frac{\lim_{n,m;n\neq m}\|x_n - x_m\|}{\lim_n \|x_n\|} \geq \frac{\frac{\epsilon+1}{2}}{1 - \Delta_{X,\beta,\tau}(\varepsilon)}$$

Finally, taking the limit as $\epsilon \to 1^-$ we deduce the desired inequality.

$\square$

Remark 18 If X is UKK(τ), from Lemma 23, $\tau CSX > 1$. Therefore, if X is a separable Banach space, τ a linear topology such that the norm is a $\tau - slcs$ function and X is UKK(τ), then has the τ-FPP [Le]. Notice that from Lemma 23 we may deduce the same conclusion considering $\Delta_{X,\beta,\tau}(1^-) > 0$.

Recall that a Banach space X is said to have the nonstrict Opial property with respect to a topology τ is

$$\liminf_n \|x_n\| \leq \liminf_n \|x + x_n\|$$

for every τ-null bounded sequence $\{x_n\}$ in X and $x \in X$. Before Lemma 15 it was proved for Hilbert spaces that $\chi(\{y_n\}) = \lim \|y_n - v\|$ if $\{y_n\}$ is w-convergent to v and $\lim \|y_n - v\|$ exists. Following the same argumente it is easy to deduce the following fact: If τ is a linear topology on X and X satisfies the nonstrict Opial property with respect to τ then $\chi(\{y_n\}) = \lim \|y_n - v\|$ if $\{x_n\}$ is a sequence in X τ-convergent to v and $\lim \|y_n - v\|$ exists. This property will be used in the proof of the next result:

LEMMA 24. *Suppose that X verifies the nonstrict Opial property with respect to a linear topology τ defined on X. Then $\tau CS(X) \geq h_0$ where*

$$h_0 = \sup\left\{t \geq 1 : \frac{1}{t} + \Delta_{X,\chi,\tau}\left(\frac{1^-}{t}\right) \geq 1\right\}.$$

PROOF. We define the function

$$h(t) = \frac{1}{t} + \Delta_{X,\chi,\tau}\left(\frac{1^-}{t}\right).$$

Since $\Delta_{X,\chi,\tau}(\cdot)$ is a nondecreasing function, h is increasing.

To obtain a contradiction, suppose that $\tau CS(X) < h_0$. Take $l \in (\tau CS(X), h_0)$ and a normalized sequence which converges to zero for the topology τ and such that $\lim_{n,m;n\neq m}\|x_n - x_m\| \leq l$.

Let ε be an arbitrary positive number with $l + 2\varepsilon < h_0$. There exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| \leq l + \varepsilon$ if $n, m \geq n_0$. Fix $k \geq n_0$ and consider the sequence

$y_n = (x_k - x_n)/(l+\varepsilon)$ with $n \geq n_0$. Then, $\{y_n\}$ is contained in the closed unit ball of X, τ-converges to the point $x_k/(l+\varepsilon)$ and

$$\left\| y_n - \frac{x_k}{l+\varepsilon} \right\| = \left\| \frac{x_n}{l+\varepsilon} \right\| = \frac{1}{l+\varepsilon} \qquad \text{for every } n \in \mathbb{N}.$$

From the nonstrict Opial condition with respect to τ we deduce $\chi(\{y_n\}) = 1/(l+\varepsilon) > 1/(l+2\varepsilon)$. Therefore, if we put $l' = l + 2\varepsilon$, we obtain:

$$\frac{1}{l'} < \frac{1}{l+\varepsilon} = \left\| \frac{x_k}{l+\varepsilon} \right\| \leq 1 - \Delta_{X,\chi,\tau}\left(\frac{1}{l+2\varepsilon}\right) = 1 - \Delta_{X,\chi,\tau}\left(\frac{1}{l'}\right).$$

Hence $h(l') < 1$, which contradicts the definition of h_0 since $l' < h_0$.

□

Remark 19

The lower bounds for $\tau CS(X)$ given in the above lemmas are the best possible in general. Notice that for the space $X = (L_p(\mu), \|\cdot\|_p)$ endowed with the *clm* topology we have:

$$\frac{1}{1 - \Delta_{X,\beta,clm}(1^-)} = 2^{1/p}, \quad h_0 = 2^{1/p}.$$

This corresponds to the actual value of $(clm)CS(L_p(\mu))$. However, the inequalities can be strict in some Banach spaces (see Chapter VI in [ADL2]).

4. Fixed points in NUS spaces

In Section 4 we studied the w-FPP as a property which is implied by either the GGLD property or normal structure. However, there are Banach spaces without these properties which have the FPP. For instance, $\ell^{p,\infty}$ does not have normal structure (see [ADL, Example VI.2]), but this space has the FPP. This fact can be proved, for instance, checking that the Banach-Mazur distance between $\ell^{p,\infty}$ and ℓ^p is $2^{1/p}$ and applying a stability result in [By3].

On the other hand, NUS spaces (see definition in Prus chapter) can also fail to have normal structure. The space $\ell^{p,\infty}$ is again an example of this assertion. However, we will prove that NUS spaces have the FPP.

We are going to prove a basic result in metric fixed point theory: Goebel-Karlovitz's lemma. We need some previous definitions.

Definition 8. *Let (X, d) be a metric space and T a mapping from X into X. A sequence $\{x_n\}$ is called an approximated fixed point sequence for T if $d(x_n, Tx_n) \to 0$ as $n \to \infty$.*

If we assume that C is a convex bounded closed subset of a Banach space X and $T : C \to C$ a nonexpansive mapping, it is easy to prove that an approximated fixed point sequence exists in C.

Definition 9. *The asymptotic radius and centre of a sequence $\{x_n\}$ in a Banach space X will be defined by:*

$$r_a(\{x_n\}, B) = \inf\{\limsup_{n\to\infty} \|x_n - y\| : y \in B\},$$
$$Z_a(\{x_n\}, B) = \{y \in B : \limsup_{n\to\infty} \|x_n - y\| = r_a(\{x_n\}, B)\},$$

for a subset B of X. When $B = \overline{co}(\{x_n\})$ we will denote $r_a(\{x_n\}, co(\{x_n\}))$ and $Z_a(\{x_n\}, co(\{x_n\}))$ respectively by $r_a(\{x_n\})$ and $Z_a(\{x_n\})$.

PROPOSITION 25. *Let K be a weakly compact convex subset of a Banach space X, and $T : K \to K$ be a nonexpansive mapping. Assume that K is minimal for T, that is, no closed convex bounded proper subset of K is invariant for T. If $\{x_n\}$ is an approximated fixed point sequence in K, then*

$$Z_a(\{x_n\}, K) = K.$$

PROOF. Let

$$Z_{a,\varepsilon}(\{x_n\}, K) = \{y \in K : \limsup_{n\to\infty} \|x_n - y\| \leq r_a(\{x_n\}, K) + \varepsilon\}.$$

It is easy to check that $Z_{a,\varepsilon}(\{x_n\}, K)$ is nonempty, closed, convex and invariant for T. Thus $Z_{a,\varepsilon}(\{x_n\}, K) = K$ and $Z_a(\{x_n\}, K) = \bigcap_{\varepsilon>0} Z_{a,\varepsilon}(\{x_n\}, K) = K$.

□

LEMMA 26 (GOEBEL-KARLOVITZ). *Let K be a weakly compact convex subset of a Banach space X, and $T : K \to K$ be a nonexpansive mapping. Assume that K is minimal for T and $\{x_n\}$ is an approximated fixed point sequence for T. Then*

$$\lim_{n\to\infty} \|y - x_n\| = \ diam(K)$$

for every $y \in K$.

PROOF. Since $T(K) \subset K$ we have $\overline{\text{co}}(T(K)) \subset K$. Thus $\overline{\text{co}}(T(K))$ is a weakly compact convex subset of K which is also invariant under T. The minimality of K implies $K = \overline{\text{co}}(T(K))$. Since K is a weakly compact and convex set we know that $Z(K)$ is a nonempty set. Let $x \in Z(K)$. For every $y \in K$ we have $\|Ty - Tx\| \leq \|y - x\| \leq r(K)$. Thus $T(K)$ is contained in the closed ball $\overline{B}(Tx, r(K))$ which implies that $\overline{\text{co}}(T(K)) = K \subset \overline{B}(Tx, r(K))$. Hence $\sup\{\|y - Tx\| : y \in K\} \leq r(K)$ which means $Tx \in Z(K)$. Thus $Z(K)$ is a convex weakly compact subset of K and is invariant under T. Again the minimality of K implies $Z(K) = K$ and so K is a diametral set.

We claim now that $\limsup_{n\to\infty} \|y - x_n\| = \ \text{diam}(K)$ for every $y \in K$. Indeed, we assume that a vector $y \in K$ exists such that $\limsup_{n\to\infty} \|y - x_n\| < \ \text{diam}(K)$. We let $r = \limsup_{n\to\infty} \|y - x_n\|$ and $d = \ \text{diam}(K)$ and consider the family $\{\overline{B}(z, (r + d)/2) \cap K : z \in K\}$. Choose any positive $\varepsilon < (d - r)/2$. From Proposition 25 we know that $\limsup_{n\to\infty} \|x_n - z\| = r$ for any $z \in K$. Thus, for any finite subset $\{z_1, ..., z_k\}$ of K an integer N exists such that $\|x_N - z_i\| \leq r + \varepsilon = (r + d)/2$ for $i = 1, ..., k$. Hence x_N belongs to $\bigcap_{i=1}^k \overline{B}(z_i, (r + d)/2)$. The weak compactness of K implies the existence of $x_0 \in \bigcap_{z\in K} \overline{B}(z, (r + d)/2) \cap K$ and this point is not diametral because

$$\sup_{z\in K} \|z - x_0\| < \frac{r + d}{2} < d = \ \text{diam}(K).$$

This contradiction proves the claim. If $\liminf_{n\to\infty} \|y - x_n\| < \ \text{diam}(K)$ for some $y \in K$ there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty} \|y_n - y\| = \liminf_{n\to\infty} \|x_n - y\| < \ \text{diam}(K)$, contradicting the claim, because every subsequence of an approximated fixed point sequence is also an approximated fixed point sequence.

□

LEMMA 27. *Let K be a weakly compact convex subset of a Banach space X, and $T : K \to K$ be a nonexpansive mapping. Assume that K is minimal for T, diam(K) $= 1$ and $\{x_n\}$ is an approximated fixed point sequence for T which is weakly null. Then, for every $\varepsilon > 0$, there exists a sequence $\{z_n\}$ in K such that:*

(i) $\{z_n\}$ is weakly convergent to a point $z \in K$.

(ii) $\|z_n\| > 1 - \varepsilon$ for every $n \in \mathbb{N}$.

(iii) $\|z_n - z_m\| \leq 1/2$ for every $n, m \in \mathbb{N}$.

(iv) $\limsup_n \|z_n - x_n\| \leq 1/2$.

PROOF. If $\{w_n\}$ is an approximated fixed point sequence in K, we know from lemma 26 that $\lim_n \|w_n\| = 1$ because we supposed that $0 \in K$. Therefore, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|x\| > 1 - \varepsilon$ if $x \in K$ and $\|Tx - x\| < \delta(\varepsilon)$. Indeed, otherwise, there exists $\varepsilon_0 > 0$ such that we can find $w_n \in K$ with $\|Tw_n - w_n\| < 1/n$ and $\|w_n\| \leq 1 - \varepsilon_0$ for every $n \in \mathbb{N}$. Thus, the sequence $\{w_n\}$ is approximated fixed point sequence in K which satisfies $\limsup_n \|w_n\| \leq 1 - \varepsilon_0$.

Let $\varepsilon > 0$. We choose $\gamma < \min\{1, \delta(\varepsilon)\}$ and for every $n \in N$ we define the contractive mapping $S_n : K \to K$ by

$$S_n(x) = (1 - \gamma)T(x) + \gamma/2x_n.$$

Banach's Contraction Principle assures the existence of a (unique) fixed point z_n for S_n. Since K is weakly compact, we can assume, taking a subsequence if necessary that $\{z_n\}$ satisfies (i). Since

$$\|z_n - Tz_n\| \leq \gamma \|Tz_n - \frac{1}{2}x_n \leq \gamma\delta(\varepsilon)$$

we know that $\{z_n\}$ satisfies (ii).

Condition (iii) easily follows and (iv) is a consequence of the inequalities

$$\|z_n - x_n\| \leq \|(1 - \gamma)Tz_n + \frac{\gamma x_n}{2} - x_n\|$$

$$\leq (1 - \gamma)\|Tz_n - Tx_n\| + (1 - \gamma)\|Tx_n - x_n\| + \frac{\gamma\|x_n\|}{2}.$$

Thus,

$$\|z_n - x_n\| \leq \frac{1}{2} + \frac{1 - \gamma}{\gamma}\|Tx_n - x_n\|.$$

Taking limit sup as n tends to infinity we obtain (iv).

□

The following definition was given in [Ga]:

DEFINITION 10. *Let X be a Banach space. We define the coefficient*

$$R(X) = \sup\{\liminf_{n \to \infty} \|x_n + x\|\}$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq 1$ and over all weakly null sequences in $\overline{B}(0, 1)$.

THEOREM 28. *Let X be a Banach space and assume that $R(X) < 2$. Then X has the weak fixed point property.*

PROOF. If we assume that X fails to have the w-FPP, we can find a weakly compact and convex subset K of X such that $\operatorname{diam}(K) = 1$ and K is minimal invariant for a nonexpansive mapping T and an approximated fixed point sequence $\{x_n\}$ in K. We can assume that $\{x_n\}$ is weakly convergent, and by translation that $\{x_n\}$ is weakly null. Consider a sequence $\{z_n\}$ satisfying (i)-(iv) in lemma 27 Taking again a subsequence, if necessary, we can assume that $\lim_n \|z_n - z\|$ exists. Furthermore, $\lim_n \|z_n - z\| \le \limsup_n \limsup_m \|z_n - z_m\| \le 1/2$.

A positive η can be chosen such that $\eta R(X) < 1 - R(X)/2$. For a large enough n, we have $\|z_n - z\| \le 1/2 + \eta$. Furthermore $\|z\| \le \liminf_{n\to\infty} \|z_n - x_n\| \le 1/2$. Hence

$$\left\| \frac{z_n}{1/2+\eta} \right\| = \left\| \frac{z_n - z}{1/2+\eta} + \frac{z}{1/2+\eta} \right\| \le R(X).$$

Thus $\limsup_{n\to\infty} \|z_n\| \le R(X)(1/2+\eta) < 1$ which is a contradiction because $0 \in K$.

□

Remark 20: Assume $X = c_0$. It is easy to check that $R(c_0) = 2$ and so c_0 has the w-FPP although c_0 does not satisfy the GGLD property.

THEOREM 29. *Let X be an NUS Banach space. Then $R(X) < 2$.*

PROOF. Since X is NUS we know that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any weakly null sequence in B_X and any $x \in B_X$ a positive integer k exists such that $\|x + tx_k\| \le 1 + \varepsilon t$ if $0 < t < \delta$. Hence,

$$\|x + x_k\| \le \|x + tx_k\| + (1-t)\|x_k\| \le 2 + t(1-\varepsilon).$$

Furthermore k can be chosen arbitrarily large because the same argument can be applied to the sequence $\{y_n\}$ where $y_1 = x$ and $y_n = x_{k+n}$ for $n > 1$. Hence $R(X) < 2$.

□

5. Asymptotically regular mapping

We shall study in this section the existence of fixed points for a different class of mappings, called asymptotically regular mappings. The concept of asymptotically regular mappings is due to Browder and Petryshyn [BP]. The fixed point theorems which we shall study are based upon coefficients concerning the modulus of NUC or Opial modulus.

DEFINITION 11. *Let (X, d) be a metric space. A mapping $T : M \to M$ is called asymptotically regular if*

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$$

for all $x \in M$.

Example 7: Let $T : [0,1] \to [0,1]$ be an arbitrary nonexpansive mapping. It is easy to check that $S = (I+T)/2$ is also nonexpansive. Thus

$$|S^{n+1}x - S^n x| \le \ldots \le |S^2 x - Sx| \le |Sx - x|.$$

Furthermore S is a nondecreasing function. Indeed, if $x \le y$ and $Sx > Sy$ we have $(x+Tx)/2 > (y+Ty)/2$ which implies $|Tx-Ty| \ge Tx-Ty > y-x = |x-y|$. Thus

$$1 \ge |S^{n+1}x - x| = \sum_{k=1}^{n} |S^{k+1}x - S^k x| \ge n|S^{n+1}x - S^n x|$$

which implies $|S^{n+1}x - S^n x| \le 1/n$. Then S is asymptotically regular. In fact, it can be proved (see [GK1, Theorem 9.4, page 98]) that if T is a nonexpansive mapping from a bounded convex subset C of a Banach space into C, then $T_\lambda = \lambda I + (1-\lambda)T$ is asymptotically regular for all $0 < \lambda < 1$.

We will show the possibility of using a modulus of noncompact convexity and Opial modulus to obtain fixed point theorems for asymptotically regular mappings. We need some notations: If S is a mapping from a set C into itself, then we use the symbol $|S|$ to denote the Lipschitz constant of S, that is,

$$|S| = \sup\left\{ \frac{\|Sx - Sy\|}{\|x-y\|} : x, y \in C, x \ne y \right\}.$$

Now if T is a mapping on C, we set

$$s(T) = \liminf_{n\to\infty} |T^n|.$$

THEOREM 30. *Let X be a Banach space, C a nonempty closed τ-sequentially compact subset of X, and $T : C \to C$ an asymptotically regular mapping. Assume that one of the following conditions is satisfied:*
(a) $s(T) < 1 + r_{X,\tau}(1)$
(b) $s(T) < \sqrt{\tau CS(X)}$
(c) $s(T) < 1/(1 - \Delta_{X,\beta,\tau}(1^-))$
Then T has a fixed point.

PROOF. Choose a subsequence $\{n_k\}$ of positive integers such that $s(T) = \lim_k |T^{n_k}|$. For any $x \in C$ we can construct a subsequence $\{n_k(x)\} \subset \{n_k\}$ such that $\{T^{n_k(x)}x\}_k$ is τ-convergent, say to $l(x)$, and $\lim_{l,k;l\ne k} \|T^{n_k(x)}x - T^{n_l(x)}x\|$ and $\lim_k \|T^{n_k(x)}x - l(x)\|$ exist. We construct a sequence $\{x_m\} \subset C$ and sequences of positive integers $\{n_k(x_m)\}$ in the following way:

$$x_0 \in C \text{ arbitrary}$$

$$x_m = l(x_{m-1}) = \tau - \lim_k T^{n_k(x_{m-1})} x_{m-1},$$

$$\{n_k(x_m)\}_k \subset \{n_k(x_{m-1})\}_k \text{ for all } k \ge 1.$$

By using a diagonal argument we consider the sequence of positive integers $n_i = n_i(x_i)$ for all $i \in \mathbb{N}$. In this way, we have that the sequence $\{T^{n_i}x_m\}$ is weakly convergent to x_{m+1}, $\lim_{i,j;i\ne j} \|T^{n_i}x_m - T^{n_j}x_m\|$ and $\lim_i \|T^{n_i}x_m - x_{m+1}\|$ exist for all $m \in \mathbb{N}$ and $s(T) = \lim_i |T^{n_i}|$.

We are going to prove that $\{x_m\}$ is norm convergent to a vector in C.

Denote $B_m = \lim_i \|T^{n_i}x_m - x_{m+1}\|$. Our proof is based upon the following claim:

Claim. Assume that $s(T)$ satisfies either (a) or (b) or (c). Then, there exists $\alpha < 1$ such that $B_m < \alpha B_{m-1}$ for any $m \in \mathbb{N}$.

Using this claim we can go on the proof. Firstly, we know that $\lim_m B_m = 0$. By using that the norm on X is τ-sequentially lower semicontinuous we deduce

$$\|x_m - x_{m+1}\| \le \limsup_i \|x_m - T^{n_i}x_m\| + \limsup_i \|T^{n_i}x_m - x_{m+1}\| \le$$

$$\limsup_i \left(\liminf_j \|T^{n_j}x_{m-1} - T^{n_i}x_m\|\right) + B_m \le s(T)B_{m-1} + B_m.$$

which implies that $\{x_m\}$ is a Cauchy sequence. Let $z = \lim_m x_m$. Since C is closed, $z \in C$. Furthermore

$$\|z - T^{n_i}z\| \le \|z - x_{m+1}\| + \|x_{m+1} - T^{n_i}x_m\| + \|T^{n_i}x_m - T^{n_i}z\| \le$$

$$\|z - x_{m+1}\| + \|x_{m+1} - T^{n_i}x_m\| + L_i\|x_m - z\|.$$

Taking the limit as $i \to \infty$ yields

$$\limsup_i \|z - T^{n_i}z\| \le \|z - x_{m+1}\| + B_m + s(T)\|x_m - z\| \to 0$$

as $m \to \infty$. Hence $T^{n_i}z \to z$. From the assumption on $s(T)$ there exits a positive integer k such that $|T^k| < +\infty$. If we choose a positive number ϵ, the asymptotic regularity of T implies that there exists $i_0 \in \mathbb{N}$ such that $\|T^{n_i+k}z - T^{n_i}z\| < \epsilon$ if $i \ge i_0$. We also assume $\|T^{n_i}z - z\| \le \epsilon$.

Hence, for $i \ge i_0$ we obtain

$$\|T^k z - z\| \le \|T^k z - T^{n_i+k}z\| + \|T^{n_i+k}z - T^{n_i}z\| + \|T^{n_i}z - z\| \le$$

$$(|T^k| + 1 + 1)\epsilon \to 0 \qquad \text{as } \epsilon \to 0.$$

yielding $T^k z = z$. It is easily verified by induction that $T^{ks}z = z$ for $s = 1, 2, \cdots$. Therefore

$$\|Tz - z\| = \|T^{ks+1} - T^{ks}z\| \to 0$$

as $s \to \infty$. So z is a fixed point for T in C.

It remains to prove the claim. From the definition of $\tau CS(X)$ we obtain

$$B_m \le \frac{\lim_{i,j;i\ne j} \|T^{n_i}x_m - T^{n_j}x_m\|}{\tau CS(X)}$$

The asymptotic regularity of T on C ensures that $\lim_j \|T^{n_j - l}x_m - x_m\| = \lim_j \|T^{n_j}x_m - x_m\|$ for any $l \in \mathbb{N}$. Thus

$$\lim_{i,j;i\ne j} \|T^{n_i}x_m - T^{n_j}x_m\| = \limsup_i \left(\limsup_j \|T^{n_i}x_m - T^{n_j}x_m\|\right) \le$$

$$\limsup_i |T^{n_i}| \limsup_j \|x_m - T^{n_j - n_i}x_m\| = s(T) \limsup_j \|x_m - T^{n_j}x_m\|.$$

Therefore

$$B_m \leq \frac{s(T)}{\tau CS(X)} \limsup_j \|x_m - T^{n_j} x_m\|.$$

Now, we split the proof in two parts. For (a) denote

$$\alpha = \sup\{c > 0 : r_{X,\tau}(c) \leq s(T) - 1\}.$$

We claim that $\alpha < 1$. Indeed, otherwise $\sup\{c > 0 : r_{X,\tau}(c) \leq s(T) - 1\} \geq 1$ which implies $r_{X,\tau}(c) \leq s(T) - 1$ for every $c < 1$. The continuity of $r_{X,\tau}(\cdot)$ implies $r_{X,\tau}(1) \leq s(T) - 1 < r_{X,\tau}(1)$, a contradiction. Consider the sequence

$$\{z_j\} = \left\{ \frac{T^{n_j} x_{m-1} - x_m}{B_{m-1}} \right\}_j$$

which is null convergent with respect to the topology τ and $\lim_j \|z_j\| = 1$. We can apply the definition of the uniform Opial modulus $r_{X,\tau}(\cdot)$ to the sequence $\{z_j\}$ and the vector $(x_m - T^{n_i} x_m)/B_{m-1}$. In this way we obtain

$$\liminf_j \left\| z_j + \frac{x_m - T^{n_i} x_m}{B_{m-1}} \right\| = \liminf_j \frac{\|T^{n_j} x_{m-1} - T^{n_i} x_m\|}{B_{m-1}} \geq$$

$$1 + r_{X,\tau}\left(\frac{\|T^{n_i} x_m - x_m\|}{B_{m-1}} \right).$$

Thus

$$1 + r_{X,\tau}\left(\frac{\|T^{n_i} x_m - x_m\|}{B_{m-1}} \right) \leq \frac{1}{B_{m-1}} L_i \liminf_j \|T^{n_j} x_{m-1} - x_m\| = L_i$$

which implies, using the continuity of $r_{X,\tau}(\cdot)$, that

$$1 + r_{X,\tau}\left(\frac{\limsup_i \|T^{n_i} x_m - x_m\|}{B_{m-1}} \right) \leq s(T)$$

and $\limsup_i \|T^{n_i} x_m - x_m\| \leq \alpha B_{m-1}$. Since $s(T) < 1 + r_{X,\tau}(1) \leq \tau CS(X)$ we obtain $B_m \leq \alpha B_{m-1}$.

For (b) and (c) we consider the sequence

$$\{z_j\} = \left\{ \frac{T^{n_i} x_m - T^{n_j} x_{m-1}}{\eta L_i B_{m-1}} \right\}_j$$

where $L_i = |T^{n_i}|$, $i \in \mathbb{N}$ is fixed and $\eta > 0$ is arbitrary.

Since

$$\limsup_j \|T^{n_i} x_m - T^{n_j} x_{m-1}\| \leq |T^{n_i}| \limsup_j \|x_m - T^{n_j} x_{m-1}\| = L_i B_{m-1},$$

there exists $j_0 \in \mathbb{N}$ such that $\{z_j\}_{j \geq j_0}$ lies in the unit ball of X. On the other hand, this sequence is τ-convergent to the vector

$$\frac{T^{n_i} x_m - x_m}{L_i B_{m-1}(1 + \eta)}.$$

Furthermore

$$\lim_{j,k;j\neq k}\|z_j - z_k\| = \frac{1}{(1+\eta)L_i B_{m-1}} \lim_{j,k;j\neq k}\|T^{n_j}x_{m-1} - T^{n_k}x_{m-1}\| \geq$$

$$\frac{1}{L_i(1+\eta)B_{m-1}}\tau CS(X)B_{m-1}.$$

Since

$$s(T) < \frac{1}{1+\Delta_{X,\beta,\tau}(1^-)} \leq \tau CS(X), \qquad (\star)$$

choose $i \in \mathbb{N}$ large enough such that $(1+2\eta)L_i < (1+3\eta)\tau CS(X)$. Applying the definition of the modulus $\Delta_{X,\beta,\tau}(\cdot)$ we obtain

$$\frac{\|T^{n_i}x_m - x_m\|}{(1+\eta)L_i B_{m-1}} \leq 1 - \Delta_{X,\beta,\tau}\left(\frac{\tau CS(X)}{(1+2\eta)L_i}\right) \leq 1 - \Delta_{X,\beta,\tau}\left(\frac{1}{1+3\eta}\right)$$

Letting $i \to \infty$ and $\eta \to 0$ we get $B_m \leq \alpha B_{m-1}$ with

$$\alpha = \frac{s(T)^2}{\tau CS(X)}\left(1 - \Delta_{X,\beta\tau}(1^-)\right).$$

Now notice that $\alpha < 1$ in case (a) because from the τ-sequential lower semicontinuity of the norm we derive that the function $\Delta_{X,\beta,\tau}(\cdot)$ is nonnegative and in case (b) we can use $(\star)$. $\square$

COMMENTS AND FURTHER RESULTS IN THIS CHAPTER.

(1) Lemma 26 was independently proved by Goebel [Go2] and Karlovitz [Ka2]. Theorems 28 and 29 were first proved in [Ga]. We follow a more direct proof using Lemma 27, which is taken from [LJ2].

Theorem 30 is basically proved in [DJ]. Previous results in this direction can be found in [CM,Kuc,DX].

(2) The converse of Theorem 21 is not true. Indeed, on the space $L_1([0,1])$ we introduce the equivalent norm

$$|||x|||^2 = \|x\|_1^2 + \left(\sum_{k=1}^{\infty}\frac{|x(k)|}{2^k}\right)^2$$

where $x = \sum_{k=1}^{\infty} x(k)e_k$, $\{e_k\}$ is a Schauder basis in $L_1([0,1])$ (for instance, the Haar system) and $\|\cdot\|_1$ is the usual norm on $L_1([0,1])$. Since $(L_1([0,1]),\|\cdot\|_1)$ fails to have the w-FPP [Al], $L_1[0,1]$ does not have the w-GGLD property. Then, there exists a weakly null sequence $\{x_n\}$ such that $\lim_n \|x_n\|_1 = \lim_{n,m;n\neq m}\|x_n - x_m\|_1$. It is easy to check that if $\{x_n\}$ is a weakly null sequence then

$$\lim_n \sum_{k=1}^{\infty}\frac{|x_n(k)|}{2^k} = 0.$$

Thus
$$\frac{\lim_{n,m;n\neq m}|||x_n-x_m|||^2}{\lim_n |||x_n|||^2}=\frac{\lim_{n,m;n\neq m}\|x_n-x_m\|_1^2}{\lim_n \|x_n\|_1^2}=1.$$
But $(L_1([0,1]),|||\cdot|||)$ is U.C.E.D. [DGZ, Corollary 6.9, page 66] and consequently this space has the w-FPP.

(3) The coefficient $WCS(X)$ is introduced in [By3] using a different definition:

$$WCS(X)=\inf\{\frac{\mathrm{diam}_a(\{x_n\})}{r_a(\{x_n\})}:\{x_n\}\text{ is a weakly convergent sequence which is not norm convergent}\}$$

where $\mathrm{diam}_a(\{x_n\})=\lim_k \sup\{\|x_n-x_m\|:n,m\geq k\}$ is the asymptotic diameter of $\{x_n\}$. However, both definitions are equivalent as proved in [ADL2].

(4) In some spaces, WCS(X) can be strictly greater than the lower bound obtained in Lemma 23, as the following example shows. Let X be the space ℓ^2 renormed by
$$\|(x^k)\|=\max\left\{|x^1|,\left(\sum_{k=2}^{\infty}(x^k)^2\right)^{\frac{1}{2}}\right\}.$$
If $\{x_n\}=\{(x_n^k)\}$ is a weakly convergent sequence, taking a subsequence and by translation we can assume that $\{x_n^1\}\to 0$. So $WCS(X)=WCS(\ell^2)=\sqrt{2}$. However, considering the sequence $\{x_n\}=\{(x_n^k)\}$ where $x_n^1=1$ for every $n\in\mathbb{N}$, and $x_n^k=\delta_{kn}$ for $k\geq 2$, it is clear that $\Delta_{X,\beta}(1)=0$.

(5) It is known that a Banach space X is reflexive if $N(X)>1$ ($N(X)$ is the normal structure coefficient defined in [By3]). However we can find nonreflexive Banach spaces, without the Schur property, such that $WCS(X)>1$. Indeed, let X be the space $\ell^1\times\ell^2$ with the norm $\|(x,y)\|=\sqrt{\|x\|_1^2+\|y\|_2^2}$ and let $\{(x_n,y_n)\}$ be a normalized weakly null sequence such that $\lim_{n,m;n\neq m}\|(x_n,y_n)-(x_m,y_m)\|$ exists. Since $\{x_n\}$ is weakly null and ℓ^1 has the Schur property we know $\{x_n\}$ converges to zero. Therefore
$$\lim_{n,m;n\neq m}\|(x_n,y_n)-(x_m,y_m)\|=\lim_{n,m;n\neq m}\|y_n-y_m\|$$
and $\lim_n\|y_n\|=1$. Since $WCS(\ell^2)=\sqrt{2}$ it is easy to check that $WCS(X)=\sqrt{2}$ although X is not reflexive.

(6) The following example shows that $WCS(X)$ (and so $N(X)$) can be 1 in a Banach space with normal structure: Let X be the ℓ^2-direct sum of the sequence spaces $\{\ell^n\}_{n\geq 2}$. This space is reflexive and it has normal structure because it is a UCED space. However, $WCS(X)=\inf\{WCS(\ell^n):n\geq 2\}=\inf\{2^{1/n}:n\geq 2\}$ $=1$.

(7) Instead of the coefficient $R(X)$ a uniparameter family of coefficients for any nonnegative number a are defined in [Do8] by
$$R(a,X)=\sup\{\liminf_{n\to\infty}\|x_n+x\|\}$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and over all weakly null sequences in $\overline{B}(0,1)$ such that $\lim_{n,m\,;n\neq m} \|x_n - x_m\| \leq 1$ and it is proved that X has the weak fixed point property if $R(a,X) < 1 + a$ for some a. The coefficient M(X) is defined now as $M(X)$ as

$$\sup\left\{ \frac{1+a}{R(a,X)} : a \geq 0 \right\}.$$

This coefficient is specially useful to obtain stability results of the w-FPP. In particular, $M(X)$ can be bounded from below using the modulus of k-uniform smoothness and the modulus of nearly uniform smoothness [ADL2, Chapter VII]. Using the coefficient $M(X)$ it is proved in [Do8] that X has the FPP if X is isomorphic to ℓ_2 and the Banach-Mazur distance between these spaces is less than $\sqrt{3}$. A better stability bound for ℓ_2 has been obtained in [Ln]

(8) An interesting open question is to know for which topology τ, Theorem 29 holds when the space X is $NUS(\tau)$. The proof of Lemma 5.1 is heavily based upon the weak lower sequential semicontinuity of the type null functions, i.e. assume that $\{x_n\}$ is a weakly null sequence and consider the function $\phi(x) = \limsup \|x_n - x\|$. Then $\phi(\cdot)$ is weakly lower sequentially semicontinous. It is known if this property is preserved for some other topologies, in particular for the weak star topology. In [J] it is proved that $\phi(\cdot)$ is clm-lower semicontinous in the spaces $L_p(\Omega)$ for $1 < p < +\infty$.

(9) The existence of fixed points for asymptotically regular mappings can be also studied using the coefficient $\kappa_\tau(X)$ defined in the following way:

(i) A number $b \geq 0$ is said to have the property (P_τ) with respect to

M if there exists $a > 1$ such that for all $x, y \in M$ and $r > 0$ with $\|x - y\| \geq r$ and each τ-convergent sequence $\{\xi_n\} \subset M$ for which

$\limsup_{n\to\infty} \|\xi_n - x\| \leq ar$ and $\limsup_{n\to\infty} \|\xi_n - y\| \leq br$, there exists some $z \in M$ such that $\liminf_{n\to\infty} \|\xi_n - z\| \leq r$.

(ii) $\kappa_\tau(M) = \sup\{b > 0 : b$ has property (P_τ) with respect to $M\}$.

(iii) $\kappa_\tau(X) = \inf\{\kappa_\tau(M) : M$ is as above $\}$.

The following result is proved in [DX]:

THEOREM. *Suppose X is a Banach space and C is a τ-sequentially compact subset of X. If $T : C \to C$ is an asymptotically regular mapping and $s(T) < \kappa_w(C)$, then T has a fixed point.*

Connections between $\kappa_\tau(X)$ and either moduli of noncompact convexity or the coefficient $\tau CS(X)$ can found in [J]

(10) The modulus of noncompact convexity corresponding to the Hausdorff measure of noncompactness can be also used to to obtain fixed point for asymptotically regular mappings. Indeed, in [DJ] the following result is proved: Let X be a Banach space, C a nonempty weakly compact subset of X, and $T : C \to C$ an asymptotically regular mapping and $h = \sup\{t \geq 1 : \frac{1}{t} + \Delta_{X,\chi}\left(\frac{1}{t}\right) \geq 1\}$. If X satisfies the non-strict Opial condition and $s(T) < h$, then T has a fixed point. On the other hand, Opial modulus also provides a lower estimate for the coefficient $\kappa_w(X)$ above defined (see [ADL2, Chapter IX]).

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