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Iterative Solutions for Zeros of Accretive Operators

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Abstract. Two iterative schemes are designed to approach zeros of m-accretive operators in Banach spaces. The first one is a kind of contractive iteration process involving with the resolvent and the second one is an averaged iteration process of the identity and the resolvent. Strong convergence for the first scheme and weak convergence for the second scheme are proved. The second scheme is also shown to have superlinear rate of convergence.

1. Introduction

Let X be a real Banach space. A (possibly multivalued) operator A with domain $D(A)$ and range $R(A)$ in X is called *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there is $j \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j \rangle \geq 0.$$

Here J is the normalized duality map of X; i.e.,

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

An accretive operator A in X is said to be m*-accretive* if $R(I + \lambda A) = X$ for all $\lambda > 0$. Throughout the article we always assume that A is an m-accretive operator such that $0 \in R(A)$ (i.e., the inclusion $0 \in Ax$ is solvable). Let

$$F := \{x \in D(A) : 0 \in Ax\} = A^{-1}(0).$$

We use J_r and A_r to denote the *resolvent* and *Yosida's approximation* of A, respectively. Namely,

$$J_r = (I + rA)^{-1}, \quad A_r = \frac{1}{r}(I - J_r), \quad r > 0.$$

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It is well-known (cf. [1]) that $J_r : \overline{D(A)} \to \overline{D(A)}$ is nonexpansive. (Recall that a mapping $T : E \to X$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, $x, y \in E$, where E is a subset of X.) We use $F(T)$ to denote the fixed point set of T (i.e. $F(T) = \{x \in E : x = Tx\}$). It is also known that $F(J_r) = F$ and $A_r x \in A(J_r x)$, where $r > 0$, $x \in D(A)$. We identify A with its graph (i.e. $(x, y) \in A \Leftrightarrow y \in Ax$).

Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Such a φ is called *gauge*. Associated with a gauge φ is the *duality map* $J_\varphi : X \to X^*$ defined by Browder [2] as follows

$$J_\varphi(x) := \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\}, \quad x \in X^*.$$

In the case of $\varphi(t) = t$, we have $J_\varphi = J$, the normalized duality map. Notice the relation:

$$J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x), \quad x \neq 0$$

so that we can use J_φ to describe the accretiveness of A.

Recall also that a Banach space is *smooth* if the

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

exists for each $x, y \in S_X$. (Here $S_X := \{v \in X : \|v\| = 1\}$ is the unit sphere of X.) X is *uniformly smooth* if X is smooth and the limit is attained uniformly for $x, y \in S_X$, and X is *Fréchet differentiable* if X is smooth and the limit is attained uniformly in $y \in S_X$. It is known that X is smooth if and only if each duality map J_φ is single-valued, that X is Fréchet differentiable if and only if each duality map J_φ is norm-to-norm continuous in X, and that X is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of X.

Following Browder [2], we say that a Banach space X has a *weakly continuous duality map* if there exists a gauge φ such that J_φ is single-valued and weak-to-weak* sequentially continuous (i.e., if $(x_n) \subset X$, $x_n \xrightarrow{w} x$, then $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x)$). It is known that ℓ^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$. (See [4] for more details on duality maps.) Setting now

$$\Phi(t) := \int_0^t \varphi(\tau) d\tau, \quad t \ge 0,$$

one sees that Φ is a convex function and

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis. We need the subdifferential inequality:

$$\Phi(\|x + y\|) \le \Phi(\|x\|) + \langle y, j_{x+y} \rangle \quad \forall x, y \in X,\ \forall j_{x+y} \in J_\varphi(x + y).$$

For a smooth X we have

$$\Phi(\|x + y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x + y) \rangle \quad \forall x, y \in X; \tag{1.1}$$

or considering the normalized duality mapping J, we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle \quad \forall x, y \in X.$$

Recall that a Banach space X satisfies *Opial's property* [11] if

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \quad \text{whenever } x_n \xrightarrow{w} x,\ y \neq x.$$

We also need the concept of retraction. Let D, E be subsets of a Banach space X such that $D \subset E$. A map $Q : E \to D$ is a *retraction* from E onto D if $Q(x) = x$ for $x \in D$. A nonexpansive retraction is a retraction which is nonexpansive. A *sunny* retraction Q is a retraction which satisfies the property:

$$Q(Qx + t(x - Qx)) = Qx, \quad x \in E,\ t \ge 0,$$

whenever $Qx + t(x - Qx) \in E$. In a smooth Banach space X, it is known (cf. [5]) that Q is a sunny nonexpansive retraction from E onto D if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx)\rangle \le 0, \quad x \in E,\ z \in D.$$

It is also known [13] (see also [5]) that if X is a uniformly smooth Banach space and D is the fixed point set of a nonexpansive self-mapping of a closed convex subset E of X, then there exists a (unique) sunny nonexpansive retraction from E onto D.

The purpose of the present article is to introduce two iterative schemes (the schemes (I) and (II) in sections 2 and 3) for approaching a zero of an m-accretive operator A is a Banach space X. The two schemes are motivated by the proximal point algorithm [14] of Rockafellar and the iterative methods of Halpern [7] and Mann [10]. In a suitable framework of Banach spaces we will show that the scheme (I) always converges strongly and the scheme (II) always converges weakly. We also show that the rate of convergence for the scheme (II) is superlinear.

2. The Iterative Scheme (I)

We now introduce the first iterative scheme. Put $C = \overline{D(A)}$. Note that C is also convex if the space X is uniformly convex (see [1]). Let (α_n) be a sequence in $[0,1]$ and (r_n) a sequence in $(0,\infty)$. Given any $u, x_0 \in C$, we define a sequence (x_n) by

$$(I) \qquad x_{n+1} := \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n \ge 0.$$

Remark 2.1. The inspiration for the iterative scheme (I) is Rockafellar's proximal point algorithm [14] for maximal monotone operators in a Hilbert space and Halpern's iteration process [7] (later developed by Lions [9], Wittmann [15] and others):

$$x_{n+1:} = \alpha_n u + (1-\alpha_n) T x_n, \quad n \ge 0,$$

where T is a nonexpansive self-mapping of a closed bounded convex subset of a Hilbert space.

Remark 2.2. Kamimura-Takahashi [8] studied a perturbed version of the scheme (I) in a Hilbert space H (with $D(A) = H$) as follows:

$$\begin{cases} x_0 &= u \in H, \\ y_n &\approx J_{r_n} x_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n, \end{cases}$$

where A is a maximal monotone operator in H and the perturbations y_n satisfy the condition:

$$\|y_n - J_{r_n} x_n\| \le \delta_n, \quad \sum_{n=0}^{\infty} \delta_n < \infty. \tag{2.2}$$

In our scheme (I), the initial datum x_0 is not necessarily the u in the iteration process. It is not hard to see that if the (unperturbed) scheme (I) converges, then the perturbed scheme also converges (to the same limit) under the perturbation criterion (2). Indeed, if (x_n) and $(\bar{x}_n)$ denote the respective sequences generated by the scheme (I) and the perturbed scheme with the same initial x_0, then we have $\|x_{n+1} - \bar{x}_{n+1}\| = (1 - \alpha_n)\|J_{r_n} x_n - y_n\| \le \delta_n \to 0$ as $n \to \infty$.

Before stating the main result of this section, we include two lemmas.

Lemma 2.1. *Let real sequences (α_n), (β_n) satisfy the conditions:*

(i) $(\alpha_n) \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently,* $\prod_{n=0}^{\infty}(1-\alpha_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1-\alpha_k) = 0$.

(ii) $\limsup_{n\to\infty} \beta_n \le 0$.

Assume a nonnegative sequence $\{s_n\}$ satisfies the inequality:

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n \beta_n, \quad n \ge 0. \tag{2.3}$$

Then $\lim_{n\to\infty} s_n = 0$.

Proof. For any $\varepsilon > 0$, let $N \ge 1$ be an integer big enough so that $\beta_n < \varepsilon$ for $n \ge N$. Repeatedly using (3), we obtain, for $n > N$,

$$s_{n+1} \le \left(\prod_{k=N}^{n} (1-\alpha_k) \right) s_N + \left(1 - \prod_{k=N}^{n} (1-\alpha_k) \right) \varepsilon.$$

Condition (i) then implies that $\limsup_{n\to\infty} s_n \le \varepsilon$. ∎

Lemma 2.2. *Assume that X has a weakly continuous duality map J_φ with gauge φ. Then A is demiclosed in the sense that A is closed in the product space $X_w \times X$, where X is equipped with the norm topology and X_w with the weak topology. That is, if $(x_n, y_n) \in A$, $x_n \xrightarrow{w} \tilde{x}$, $y_n \xrightarrow{s} \tilde{y}$, then $(x, y) \in A$.*

Proof. We first observe that A is accretive if and only if, for all $y_i \in Ax_i$ $(i = 1, 2)$,

$$\langle y_1 - y_2, J_\varphi(x_1 - x_2)\rangle \ge 0.$$

To prove that A is closed in $X_w \times X$ we assume $\{(x_n, y_n)\} \subset A$ satisfying $x_n \xrightarrow{w} \tilde{x}$, $y_n \xrightarrow{s} \tilde{y}$. For any point $(u, v) \in A$ we have

$$\langle y_n - v, J_\varphi(x_n - u)\rangle \geq 0, \quad n \geq 1. \tag{2.4}$$

Since J_φ is weakly continuous, we have $J_\varphi(x_n - u) \xrightarrow{w^*} J_\varphi(\tilde{x} - u)$. But $y_n - v \xrightarrow{s} \tilde{y} - v$, taking the limit as $n \to \infty$ in (4) we get

$$\langle \tilde{y} - v, J_\varphi(\tilde{x} - u)\rangle \geq 0 \quad \forall (u, v) \in A.$$

Hence $(\tilde{x}, \tilde{y}) \in A$ since A is $m-$accretive. ■

Theorem 2.1. *Suppose that X is a uniformly smooth Banach space with a weakly continuous duality map J_φ with gauge φ. Assume the following conditions:*

(i) $\alpha_n \to 0$, $\sum_n \alpha_n = \infty$.

(ii) $r_n \to \infty$.

Then (x_n) converges strongly to a point in F.

Proof. Since X is a uniformly smooth Banach space, there exists a unique sunny nonexpansive retraction Q from C onto F. In particular, we have

$$\langle x - Q(x), J_\varphi(v - Q(x))\rangle \leq 0, \quad x \in C, \ v \in F.$$

It is not hard to see that (x_n) is bounded, so is $(J_{r_n} x_n)$. Indeed, for a fixed $q \in F$, since J_{r_n} is nonexpansive and q is a fixed point of J_{r_n}, we have, for $n \geq 0$,

$$\begin{aligned} \|x_{n+1} - q\| &\leq \alpha_n \|u - q\| + (1 - \alpha_n)\|J_{r_n} x_n - q\| \\ &\leq \alpha_n \|u - q\| + (1 - \alpha_n)\|x_n - q\|. \end{aligned}$$

This combining with an induction shows that

$$\|x_n - q\| \leq \max\{\|u - q\|, \|x_0 - q\|\}, \quad n \geq 0.$$

Hence (x_n) is bounded. We now show

$$\limsup_{n\to\infty} \langle u - p, J_\varphi(x_n - p)\rangle \leq 0, \tag{2.5}$$

where $p = Q(u)$. To prove the above assertion we take a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{w} \tilde{x}$ and

$$\limsup_{n\to\infty} \langle u - p, J_\varphi(x_n - p)\rangle = \lim_{j\to\infty} \langle u - p, J_\varphi(x_{n_k} - p)\rangle. \tag{2.6}$$

We have by definition of the scheme

$$\|x_{n+1} - J_{r_n} x_n\| = \alpha_n \|u - J_{r_n} x_n\| \to 0 \ (n \to \infty).$$

Hence

$$J_{r_{n_j-1}}x_{n_j-1} \xrightarrow{w} \tilde{x}. \tag{2.7}$$

Now we have

$$(J_{r_{n_j-1}}x_{n_j-1}, A_{r_{n_j-1}}x_{n_j-1}) \in A, \quad J_{r_{n_j-1}}x_{n_j-1} \xrightarrow{w} \tilde{x}, \quad A_{r_{n_j-1}}x_{n_j-1} \xrightarrow{s} 0.$$

Apply Lemma 2.2 to get $(\tilde{x}, 0) \in A$, i.e., $\tilde{x} \in A^{-1} = F$. From (5) and the weak continuity of J_φ it follows that

$$\limsup_{n\to\infty} \langle u - p, J_\varphi(x_n - p)\rangle = \langle u - p, J_\varphi(\tilde{x} - p)\rangle \leq 0$$

since Q is sunny.

Finally we prove that $x_n \xrightarrow{s} \tilde{x}$. Use the subdifferential inequality (1) to get

$$\begin{aligned}
\Phi(\|x_{n+1} - p\|) &= \Phi(\|(1-\alpha_n)(J_{r_n}x_n - p) + \alpha_n(u-p)\|) \\
&\leq \Phi((1-\alpha_n)\|J_{r_n}x_n - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle u - p, J_\varphi(x_{n+1} - p)\rangle.
\end{aligned}$$

Now apply Lemma 2.1 to obtain $\Phi(\|x_n - p\|) \to 0$ and the proof is complete. ■

Remark 2.3. The proximal point algorithm [14] of Rockafellar always converges weakly (but not strongly, in general [6]). It is interesting that the scheme (I), a convex combination of the proximal point algorithm with a fixed point, always converges strongly.

3. The Iterative Scheme (II)

We next introduce the second iterative scheme. Given an initial $x_0 \in C$ define (x_n) by

$$x_{n+1} := \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n, \quad n \geq 0. \tag{II}$$

Remark 3.1. The scheme (II) is again motivated by Rockafellar's proximal point algorithm for maximal monotone operators in Hilbert spaces and by Mann's averaged iteration process [10] for real-valued functions in $\mathbf{R}^1$:

$$x_{n+1} := \alpha_n x_n + (1-\alpha_n)f(x_n), \quad n \geq 0.$$

This Mann's iteration process has extensively been studied over the last twenty years for constructions of fixed points of nonlinear mappings and of solutions of nonlinear operator equations involving monotone, accretive and pseudo-contractive operators.

Remark 3.2. If A is a maximal monotone operator in a Hilbert space H with $D(A) = H$, Kamimura-Takahasi [8] considered an perturbed scheme of the scheme (II) as follows:

$$\begin{cases}
x_0 &= u \in H, \\
y_n &\approx J_{r_n}x_n, \\
x_{n+1} &= \alpha_n x_n + (1-\alpha_n)y_n,
\end{cases}$$

where the perturbations y_n satisfy the same criterion (2):

$$\|y_n - J_{r_n}x_n\| \le \delta_n, \quad \sum_{n=0}^{\infty} \delta_n < \infty.$$

It is easy to see that, under the above perturbation criterion, the convergence of the scheme (II) and the perturbed version is equivalent.

In order to prove the convergence of the scheme (II) we need the inequality characterization of uniform convexity due to Xu [16]. First we recall that a Banach space X is *uniformly convex* if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$, where $\delta(\cdot)$ is the *modulus of convexity* of X defined by

$$\delta(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon\}.$$

Recall also that X is said to be *uniformly convex of power* γ (or γ*-uniformly convex*), where $\gamma \ge 2$ is a constant, if there is a constant $c_\gamma > 0$ such that $\delta(\varepsilon) \ge c_\gamma \varepsilon^\gamma$, $\varepsilon > 0$. It is well-known that both ℓ^p and L^p $(1 < p < \infty)$ are $\max\{2, p\}$-uniformly convex.

Lemma 3.1. ([16]) *Let X be a Banach space and $\gamma \ge 2$ a number. Then*

(i) X is uniformly convex if and only if, for any positive number r, there is a strictly increasing continuous function $g_r : [0, \infty) \to [0, \infty)$, $g_r(0) = 0$, such that

$$\|tx + (1 - t)y\|^\gamma \le t\|x\|^\gamma + (1 - t)\|y\|^\gamma - W_\gamma(t)g_r(\|x - y\|),$$

where $t \in [0, 1]$, $x, y \in B_r := \{u \in X : \|u\| \le r\}$, the closed ball of X centered at the origin with radius r, and $W_\gamma(t) = t^\gamma(1 - t) + t(1 - t)^\gamma$.

(ii) X is $\gamma-$uniformly convex if and only of there holds the inequality:

$$\|tx + (1 - t)y\|^\gamma \le t\|x\|^\gamma + (1 - t)\|y\|^\gamma - c_\gamma W_\gamma(t)\|x - y\|^\gamma, \quad t \in [0, 1],\ x, y \in X,$$

where $c_\gamma > 0$ is a constant.

We also need the following three lemmas.

Lemma 3.2. ([12]) *Let C be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, and let (T_n) be a sequence of nonexpansive self-mappings of C with a nonempty common fixed point set F. If $x_1 \in C$ and $x_{n+1} = T_n x_n$ for $n \ge 1$, then $\lim_{n\to\infty} \langle x_n, J(f_1 - f_2)\rangle$ exists for all $f_1, f_2 \in F$. In particular, $\langle q_1 - q_2, J(f_1 - f_2)\rangle = 0$, where $f_1, f_2 \in F$ and q_1, q_2 are weak limit points of (x_n).*

Lemma 3.3. *(The Demiclosedness Principle of Nonexpansive Mappings [3]) Let T be a nonexpansive self-mapping of a closed convex subset E of a uniformly convex Banach space. Suppose T has a fixed point. Then $I - T$ is demiclosed. This means that*

$$(x_n) \subset E,\ x_n \xrightarrow{w} x,\ (I - T)x_n \xrightarrow{s} y \quad \Rightarrow \quad (I - T)x = y.$$

Lemma 3.4. *(The Resolvent Identity [Cf. 1, 4]) For $x \in C$ and λ, $\mu > 0$,*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + (1 - \frac{\mu}{\lambda}) J_\lambda x \right).$$

Theorem 3.1. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm. Assume X has a weakly continuous duality map J_φ with gauge φ. Assume also*

(i) $\alpha_n \to 0$.

(ii) $r_n \to \infty$.

Then the scheme (II) converges weakly to a point q in F.

Proof. First we observe that for any $p \in F$, the sequence $\{\|x_n - p\|\}$ is nonincreasing. Indeed we have by nonexpansivity of J_{r_n}

$$\|x_{n+1} - p\| \le \alpha_n \|x_n - p\| + (1 - \alpha_n)\|J_{r_n} x_n - p\| \le \|x_n - p\|.$$

In particular, (x_n) is bounded, so is $(J_{r_n} x_n)$. Let $\omega_w(x_n)$ be the set of weak limit points of the sequence (x_n). Note that we can rewrite the scheme (II) in the form

$$x_{n+1} = T_n x_n, \quad n \ge 0,$$

where T_n is the nonexpansive mapping given by

$$T_n x := \alpha_n x + (1 - \alpha_n) J_{r_n} x, \quad x \in C.$$

Then we have $F(T_n) = F(J_{r_n}) = F$ for $n \ge 1$. Hence by Lemma 3.2 we get

$$\langle q_1 - q_2, J(f_1 - f_2) \rangle = 0, \quad q_1, q_2 \in \omega_w(x_n),\ f_1, f_2 \in F.$$

Therefore (x_n) will converge weakly to a point in F if we can show that $\omega_w(x_n) \subset F$. To show this we take a point v in $\omega_w(x_n)$. Then we have a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{w} v$. Noting

$$\|x_{n+1} - J_{r_n} x_n\| = \alpha_n \|x_n - J_{r_n} x_n\| \to 0,$$

we have

$$A_{r_{n_k - 1}} x_{n_k - 1} \in A J_{r_{n_k - 1}} x_{n_k - 1}, \quad A_{r_{n_k - 1}} x_{n_k - 1} \xrightarrow{s} 0, \quad J_{r_{n_k - 1}} x_{n_k - 1} \xrightarrow{w} v.$$

By Lemma 2.2 we conclude that $0 \in Av$, i.e., $v \in F$. ■

If we allow (α_n) to be away from 0 and 1, we are able to weaken the weak continuity of the duality map J_φ.

Theorem 3.2. *Let X be a uniformly convex Banach space which either has a Fréchet differentiable norm or satisfies Opial's property. Assume for some $\varepsilon > 0$,*

(i) $\varepsilon \le \alpha_n \le 1 - \varepsilon$ *for* $n \ge 1$.

(ii) $r_n \geq \varepsilon$ *for* $n \geq 1$.

Then the scheme (II) *converges weakly to a point* q *in* F.

Proof. We have shown that $\lim_n \|x_n - p\|$ exists for all $p \in F$. Applying Lemma 3.1(i) we have a strictly increasing continuous function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that (for a big enough $r > 0$)

$$\begin{aligned} \|x_{n+1} - p\|^2 \quad &\leq \quad \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|J_{r_n} x_n - p\|^2 \\ &\quad\ -\alpha_n(1 - \alpha_n) g(\|x_n - J_{r_n} x_n\|). \end{aligned}$$

This implies that

$$\alpha_n(1 - \alpha_n) g(\|x_n - J_{r_n} x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{3.8}$$

Since $\alpha_n(1 - \alpha_n) \geq \varepsilon^2$, we obtain by (8)

$$\sum_n g(\|x_n - J_{r_n} x_n\|) < \infty \quad \Rightarrow \quad \lim_n \|x_n - J_{r_n} x_n\| = 0. \tag{3.9}$$

For any fixed $\lambda \in (0, \varepsilon)$, by Lemma 3.4, we have

$$J_{r_n} x_n = J_\lambda \left(\frac{\lambda}{r_n} x_n + \left(1 - \frac{\lambda}{r_n}\right) J_{r_n} x_n \right).$$

We deduce that

$$\begin{aligned} \|J_{r_n} x_n - J_\lambda x_n\| \quad &\leq \quad \left\| \left(\frac{\lambda}{r_n} x_n + \left(1 - \frac{\lambda}{r_n}\right) J_{r_n} x_n \right) - x_n \right\| \\ &= \quad \left(1 - \frac{\lambda}{r_n}\right) \|x_n - J_{r_n} x_n\| \\ &\leq \quad \|x_n - J_{r_n} x_n\| \ \to 0 \ (n \to \infty). \end{aligned}$$

Therefore we obtain by (9),

$$\|x_n - J_\lambda x_n\| \ \to 0 \ (n \to \infty), \ \lambda \in (0, \varepsilon).$$

Apply Lemma 3.3 to find out that $\omega_w(x_n) \subset F(J_\lambda) = F$. It remains to show that $\omega_w(x_n)$ is a singleton set. Towards this end we take $p, q \in \omega_w(x_n)$ and distinguish two cases.

1°: X has a Fréchet differentiable norm.

2°: X satisfies Opial's condition.

In case 1° we apply Lemma 3.2 to get $\langle p - q, J(p - q) \rangle = 0$; hence $p = q$. In case 2° we can find two subsequences (x_{n_i}), (x_{m_j}) such that $x_{n_i} \xrightarrow{w} p$, $x_{m_j} \xrightarrow{w} q$. If $p \neq q$, Opial's property creates the contradiction:

$$\begin{aligned}\lim_n \|x_n - p\| &= \lim_i \|x_{n_i} - p\| \\ &< \lim_i \|x_{n_i} - q\| \\ &= \lim_j \|x_{m_j} - q\| \\ &< \lim_i \|x_{m_j} - p\| \\ &= \lim_n \|x_n - p\|.\end{aligned}$$

In either case we have shown that $\omega_w(x_n)$ consists of exact one point which is clearly the weak limit of (x_n). ■

Remark 3.3. If X is a Hilbert space H, Kamimura-Takahashi [8] proved that (1°) the weak limit q of (x_n) in Theorem 3.1 is actually the strong limit of the sequence (Px_n), where P is the metric projection of H onto F, and (2°) the condition (i) in Theorem 3.1 can be relaxed to the condition: $(\alpha_n) \subset [0,k]$ for some constant $k \in [0,1)$. We can prove that the same holds true if X is a γ-uniformly convex Banach space.

Theorem 3.3. *Let (x_n) be the sequence of iterations generated by the scheme (II). Then*

(i) If X is a uniformly convex Banach, $s-\lim_{n\to\infty} Px_n$ exists, where P is the metric projection of X onto F.

(ii) If X is a uniformly convex space satisfying Opial's property and if (x_n) weakly converges, then $s-\lim_{n\to\infty} Px_n = w-\lim_{n\to\infty} x_n$.

(iii) If X is $\gamma-$uniformly convex for some $\gamma \geq 2$ with a Fréchet differentiable norm and if X has a weakly continuous duality map J_φ with gauge φ, then the condition (i) in Theorem 3.1 can be weakened to the condition: $(\alpha_n) \subset [0,k]$ for some $k \in [0,1)$.

Proof. To prove (i) first we note that since F is a closed convex subset of the uniformly convex Banach space X, the metric projection P from X onto F is well defined. Next we observe that the sequence $\{\|Px_n - x_n\|\}$ is nonincreasing. Indeed we have

$$\begin{aligned}\|x_{n+1} - Px_{n+1}\| &\leq \|x_{n+1} - Px_n\| \\ &\leq \alpha_n\|x_n - Px_n\| + (1-\alpha_n)\|J_{r_n}x_n - Px_n\| \\ &\leq \|x_n - Px_n\|.\end{aligned}$$

Let $\tau := \lim_{n\to\infty}\|Px_n - x_n\|$. If $\tau = 0$, then noting

$$\begin{aligned}\|Px_n - Px_m\| &\leq \|Px_n - x_{n+m+1}\| + \|Px_m - x_{n+m+1}\| \\ &\leq \|Px_n - x_n\| + \|Px_m - x_m\| \ \to 0 \ (n,m\to\infty),\end{aligned}$$

we immediately see that (Px_n) is Cauchy and hence convergent. So assume $\tau > 0$. Put

$$\sigma := \limsup_{n,m\to\infty} \|Px_n - Px_m\|.$$

If $\sigma > 0$, then we can take $\varepsilon > 0$ small enough so that

$$(\tau+\varepsilon)\left[1-\delta\left(\frac{\sigma}{\tau+\varepsilon}\right)\right] < \tau,$$

where δ is the modulus of convexity of X. Let $N \geq 1$ be an integer such that $\|Px_n - x_n\| < \tau + \varepsilon$ for all $n \geq N$. Then since $\|x_{n+m+1} - Px_n\| \leq \|x_n - Px_n\| < \tau + \varepsilon$ and $\|x_{n+m+1} - Px_m\| \leq \|x_m - Px_m\| < \tau + \varepsilon$ for $n, m \geq N$, it follows that for $n, m \geq N$,

$$\begin{aligned}
\tau &\leq \|x_{n+m+1} - Px_{n+m+1}\| \\
&\leq \left\|x_{n+m+1} - \frac{1}{2}(Px_n + Px_m)\right\| \\
&\leq (\tau+\varepsilon)\left[1-\delta\left(\frac{\|Px_n - Px_m\|}{\tau+\varepsilon}\right)\right].
\end{aligned}$$

Letting $n, m \to \infty$ in the last inequality, we arrive at the contradiction:

$$\tau \leq (\tau+\varepsilon)\left[1-\delta\left(\frac{\sigma}{\tau+\varepsilon}\right)\right] < \tau.$$

We therefore must have that $\sigma = 0$ and (Px_n) is strongly convergent. Let v be the limit of (Px_n). It is easily seen that $\lim \|x_n - v\| = \min\{\lim \|x_n - z\| : z \in F\}$.

To prove (ii) we let $q \in F$ be the weak limit of (x_n). Remember that we have shown that $\lim_n \|x_n - y\|$ exists for each $y \in F$. We have

$$\lim \|x_n - v\| = \lim \|x_n - Px_n\| \leq \lim \|x_n - q\|.$$

So, if the space X has Opial's property, we must have $v = q$. This finishes the proof of (ii). To prove (iii) all we need to do is this:

$$\lim_{n\to\infty} \|x_{n+1} - J_{r_n}x_n\| = 0. \tag{3.10}$$

Since X is $\gamma-$uniformly convex, apply Lemma 3.1(ii) to get for $p \in F$,

$$\begin{aligned}
\|x_{n+1} - p\|^\gamma &= \|\alpha_n(x_n - p) + (1-\alpha_n)(J_{r_n}x_n - p)\|^\gamma \\
&\leq \alpha_n\|x_n - p\|^\gamma + (1-\alpha_n)\|J_{r_n}x_n - p\|^\gamma \\
&\quad -c_\gamma W_\gamma(\alpha_n)\|x_n - J_{r_n}x_n\|^\gamma \\
&\leq \|x_n - p\|^\gamma - c_\gamma W_\gamma(\alpha_n)\|x_n - J_{r_n}x_n\|^\gamma.
\end{aligned}$$

It follows that

$$[\alpha_n^\gamma(1-\alpha_n) + \alpha_n(1-\alpha_n)^\gamma]\|x_n - J_{r_n}x_n\|^\gamma \leq c_\gamma^{-1}(\|x_n - p\|^\gamma - \|x_{n+1} - p\|^\gamma) \to 0.$$

This implies that $\alpha_n^\gamma\|x_n - J_{r_n}x_n\|^\gamma \to 0$ for $0 \leq \alpha_n \leq k < 1$ for all n. Since $x_{n+1} - J_{r_n}x_n = \alpha_n(x_n - J_{r_n}x_n)$, we see that (10) holds. Repeating the last part of the proof of Theorem 3.1, we conclude that (x_n) is weakly convergent to a point in F. ■

4. Rate of Convergence

We shall show that the scheme (II) has a superlinear rate of convergence. Let (x_n) be the sequence generated by the scheme (II). Recall that (x_n) converges to $\tilde{x}$ at a *superlinear rate* of convergence if

$$\lim_{n\to\infty} \frac{\|x_{n+1}-\tilde{x}\|}{\|x_n-\tilde{x}\|} = 0.$$

Following Rockafellar [14] we say that A^{-1} is *Lipschitz continuous* at 0 with modulus $a > 0$ if there is a unique solution $\tilde{x}$ to $0 \in A(x)$ (i.e. $A^{-1}(0) = \{\tilde{x}\}$), and for some $\tau > 0$ we have

$$\|x-\tilde{x}\| \le a\|y\| \quad \text{whenever } x \in A^{-1}(y) \text{ and } \|y\| \le \tau.$$

To prove the superlinear rate of convergence of the scheme (II) we need the following inequality in a $\gamma-$uniformly convex Banach space.

Lemma 4.1. ([16]) *Assume X is a $\gamma-$uniformly convex Banach space for some $\gamma \ge 2$. Then there is a constant $c > 0$ such that*

$$\|x+y\|^\gamma \ge \|x\|^\gamma + \gamma\langle y, j_\gamma(x)\rangle + c\|y\|^\gamma \quad \forall x, y \in X,$$

where $j_\gamma(x) \in J_\varphi(x)$ and J_φ is the duality map with gauge $\varphi(t) := t^{\gamma-1}$.

Theorem 4.1. *Let (x_n) be the sequence generated by the iterative scheme (II). Let the assumptions of Theorem 3.1 be satisfied. Assume further that X is $\gamma-$uniformly convex with some $\gamma \ge 2$. Assume also A^{-1} is Lipschitz continuous at 0 with modulus $a > 0$. Then (x_n) is strongly convergent to $\tilde{x} = A^{-1}(0)$ superlinearly. More precisely we have for large enough n,*

$$\|x_{n+1}-\tilde{x}\| \le \left[\alpha_n + \frac{a(1-\alpha_n)}{(a^\gamma + cr_n^\gamma)^{1/\gamma}}\right]\|x_n-\tilde{x}\|.$$

Proof. We have shown that both (x_n) and $(J_{r_n}x_n)$ are bounded. Hence $\|A_{r_n}x_n\| = r_n^{-1}\|x_n - J_{r_n}x_n\| \to 0$. Since A^{-1} is Lipschitz continuous at 0 with modulus $a > 0$, we have a positive integer N such that

$$\|J_{r_n}x_n - \tilde{x}\| \le a\|A_{r_n}x_n\| \quad n \ge N. \tag{4.11}$$

On the other hand, since for $x, z \in D(A)$ and $r > 0$, $x - z = (J_rx - J_rz) + r(A_rx - A_rz)$. Apply Lemma 4.1 to get

$$\begin{aligned}\|x-z\|^\gamma &\ge \|J_rx - J_rz\|^\gamma + cr^\gamma\|A_rx - A_rz\|^\gamma \\ &\quad + \gamma\langle r(A_rx - A_rz), J_\varphi(J_rx - J_rz)\rangle \\ &\ge \|J_rx - J_rz\|^\gamma + cr^\gamma\|A_rx - A_rz\|^\gamma\end{aligned}$$

since $\langle A_rx - A_rz, J_\varphi(J_rx - J_rz)\rangle \ge 0$ for A is accretive. In particular,

$$\|x_n-\tilde{x}\|^\gamma \ge \|J_{r_n}x_n - \tilde{x}\|^\gamma + cr_n^\gamma\|A_{r_n}x_n\|^\gamma. \tag{4.12}$$

Combining (11) and (12) we get

$$\|J_{r_n}x_n - \tilde{x}\| \le \frac{a}{(a^\gamma + cr_n^\gamma)^{1/\gamma}}\|x_n - \tilde{x}\|. \tag{4.13}$$

It now follows from the scheme (II) and (13) that for $n \ge N$,

$$\|x_{n+1} - \tilde{x}\| \le \left[\alpha_n + \frac{a(1-\alpha_n)}{(a^\gamma + cr_n^\gamma)^{1/\gamma}}\right]\|x_n - \tilde{x}\|.$$

Since $\alpha_n \to 0$ and $r_n \to \infty$, we have $x_n \xrightarrow{s} \tilde{x}$ and $\lim_n \|x_{n+1} - \tilde{x}\|/\|x_n - \tilde{x}\| = 0$. ■

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