# SOME THEOREMS ON THE STABILITY OF MULTIBREATHERS IN KLEIN-GORDON LATTICES

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# Klein–Gordon systems with linear coupling

## Dynamical equations

$$\ddot{u}_n + V'(u_n) + \varepsilon \sum_{m=1}^{N} C_{n\,m} u_m = 0 \quad n = 1, \ldots, N$$

**Coupling examples** ($\varepsilon > 0$):

- Elastic attractive coupling: $\sum_{m=1}^{N} C_{n\,m} u_m = 2\,u_n - u_{n+1} - u_{n-1}$
- Next–neighbor, dipole–dipole repulsive coupling: $\sum_{m=1}^{N} C_{n\,m} u_m = u_{n+1} + u_{n-1}$
- Dipolar long–range interaction repulsive coupling: $\sum_{m=1}^{N} C_{n\,m} u_m = \sum_{m=1}^{N} \frac{u_m}{r^3_{n\,m}}$

**Notation** $|u\rangle = \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t)] \end{bmatrix}$ ; $|V(u)\rangle = \begin{bmatrix} V(u_1) \\ \vdots \\ V(u_N) \end{bmatrix}$ ; $|\dot{u}\rangle = \begin{bmatrix} \frac{\mathrm{d}u}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}u_N}{\mathrm{d}t} \end{bmatrix}$ ; $|V'(u)\rangle = \begin{bmatrix} \frac{\partial V}{\partial u_1} \\ \vdots \\ \frac{\partial V}{\partial u_N} \end{bmatrix}$

## Linear stability and Newton operator

- Dynamical equation: $|\ddot{u}\rangle + |V'(u)\rangle + \varepsilon\, C\, |u\rangle = 0$
  $|u\rangle$ time–reversible and periodic with frequency $\omega_{\rm b} = \dfrac{2\pi}{T}$
- (Linear) stability equation

$$\mathcal{N}_\varepsilon(u)\,|\xi\rangle \equiv |\ddot{\xi}\rangle + V''(u)\cdot|\xi\rangle + \varepsilon\, C\,|\xi\rangle = E\,|\xi\rangle$$

  Newton operator: $\mathcal{N}_\varepsilon$
- The Floquet Matrix: $\begin{bmatrix} \{\xi_n(T)\} \\ \{\dot{\xi}_n(T)\} \end{bmatrix} = \mathcal{F}_E \begin{bmatrix} \{\xi_n(0)\} \\ \{\dot{\xi}_n(0)\} \end{bmatrix}$
- Floquet multipliers of $\mathcal{F}_E$ and arguments:

$$\lambda_i = \exp({\rm i}\theta_i) \quad ; \quad i = 1, \dots, 2\,N$$

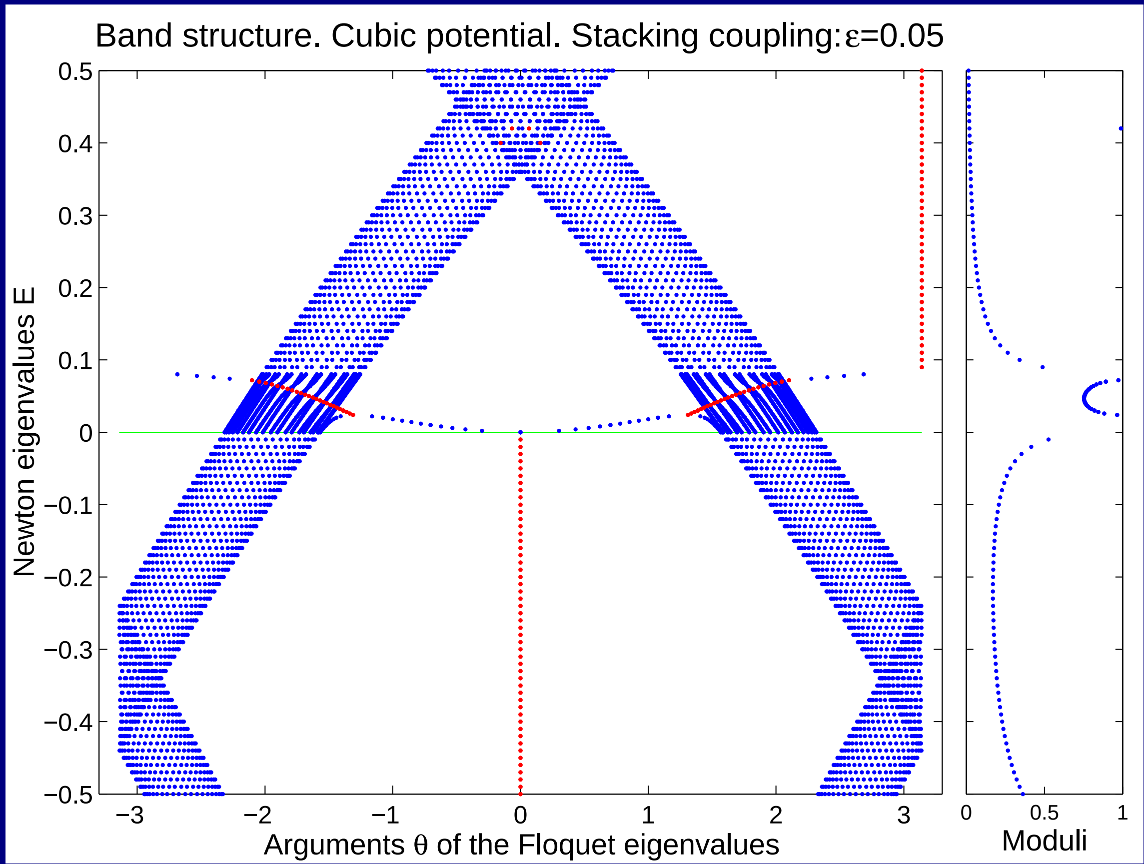
- Band structure

$$(\{\theta_l\}, E) \quad \text{with} \quad \theta_l \in \mathcal{R}$$

- Stability:
  *The solution $|u\rangle$ is linearly stable if there are 2N band intersections or tangent points with their multiplicity with the axis $E = 0$.*
  S Aubry. *Physica D*, 103:201–250, 1997.

# Example of bands

Cubic potential, attractive elastic coupling, $\varepsilon = 0.05$.
Stable single breather.




Bands figures from:
A Alvarez, JFR Archilla, J Cuevas and FR Romero, *Dark breathers in Klein-Gordon lattices. Band analysis of their stability properties.* New Journal of Physics 4:72.1-72.19 (2002).

# Bands at the anticontinuous limit ($\varepsilon = 0$)

**Stability equation:** $\mathcal{N}_0(u)\,|\xi\rangle \;\equiv\; |\ddot{\xi}\rangle \;+\; V''(u)\cdot|\xi\rangle \;=\; E\,|\xi\rangle\,,$

- $N-p$ excited oscillators . Code $\sigma_n = \pm 1$

$$\ddot{\xi}_n \;+\; V''(u_n)\,\xi_n \;=\; E\,\xi_n$$

With $E=0$:
Phase mode (periodic): $\dot{u}_n$
Growth mode (unbounded): $\dfrac{\partial u_n}{\partial \omega_\mathrm{b}}$

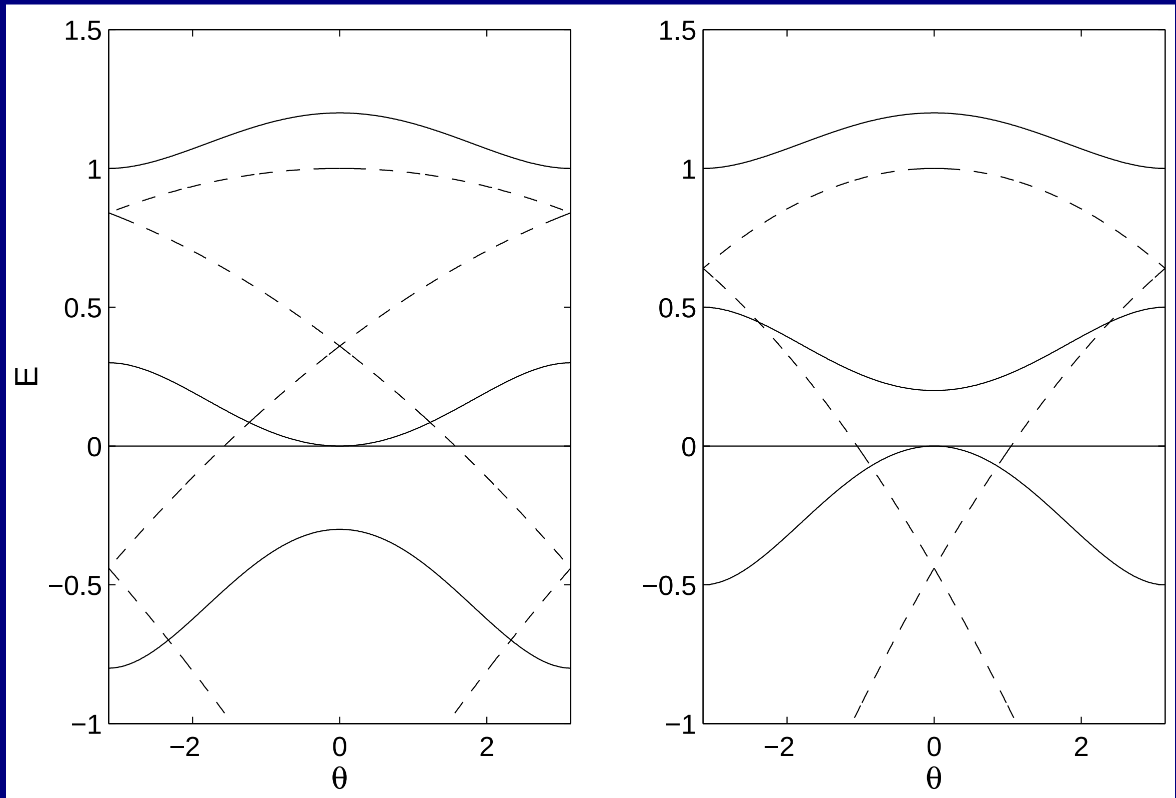
- $p$ rest oscillators. Code $\sigma_n = 0$

$$\ddot{\xi}_n \;+\; (\omega_0)^2 \xi_n \;=\; E\,\xi_n \quad ; \quad \omega_0 = \sqrt{V''(0)}$$

**Bands**

- Excited oscillators $(N-p)$: Tangent from above (soft) or below (hard)
- Rest oscillators $(p)$: $E = \omega_0^2 - \omega_\mathrm{b}^2 (\frac{\theta}{2\pi})^2$

# Bands at the anticontinuous limit

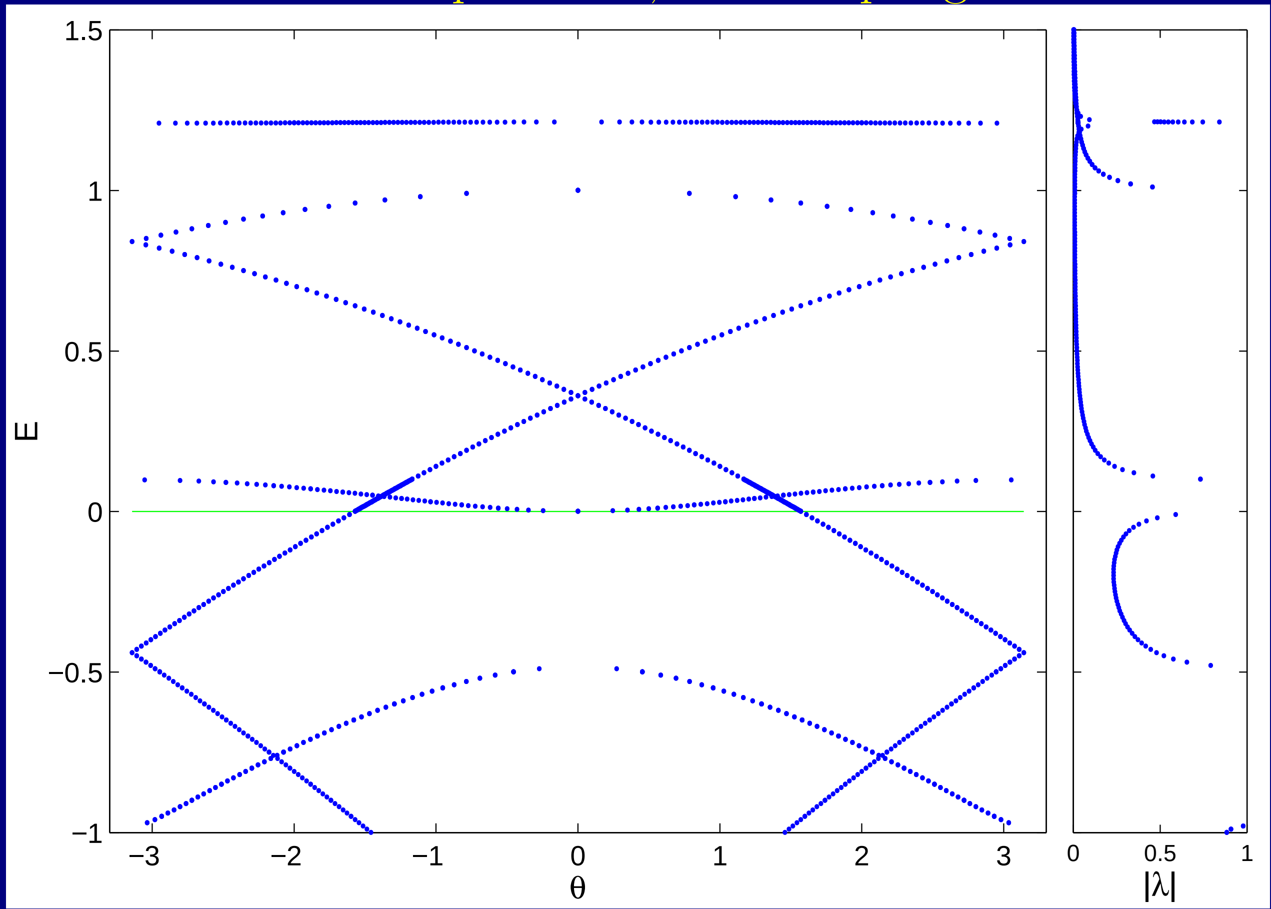



Soft Hard

- - - - Rest oscillators —— Excited oscillators

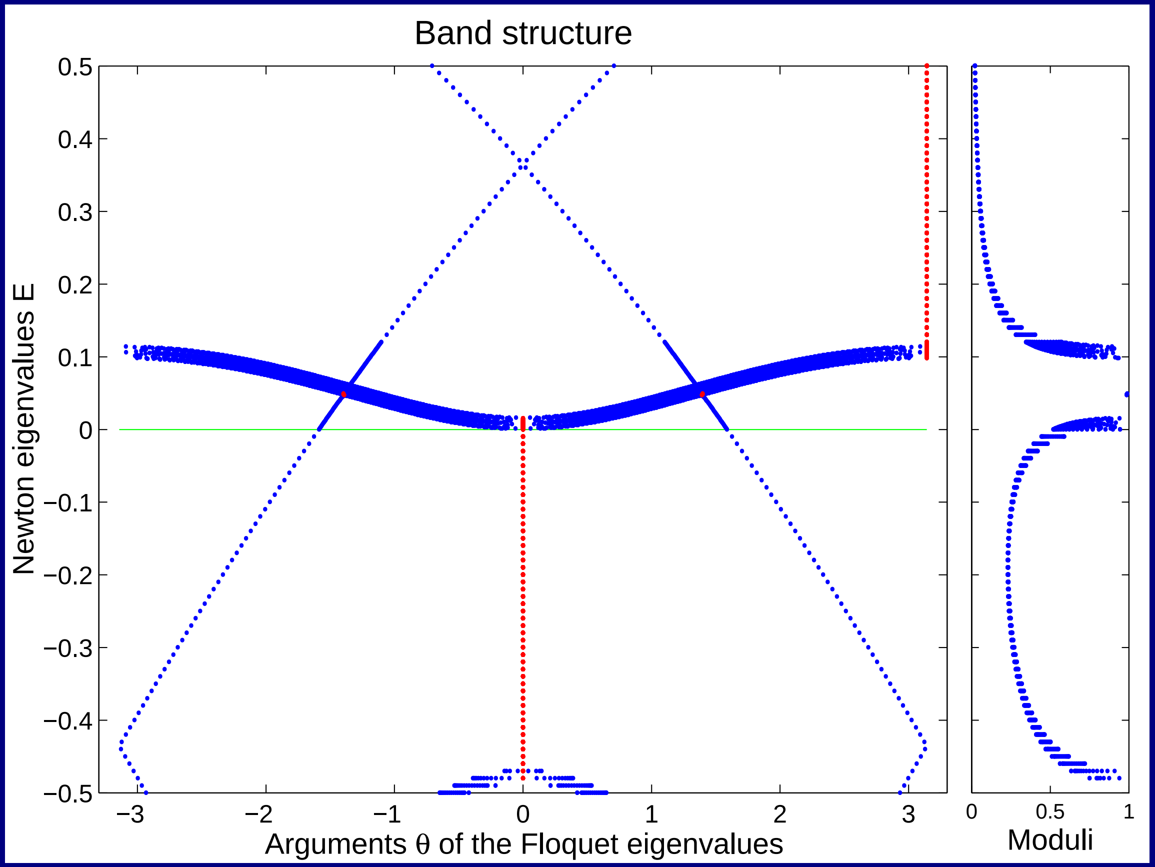
# Example of bands at $\varepsilon = 0$

Cubic potential, zero coupling



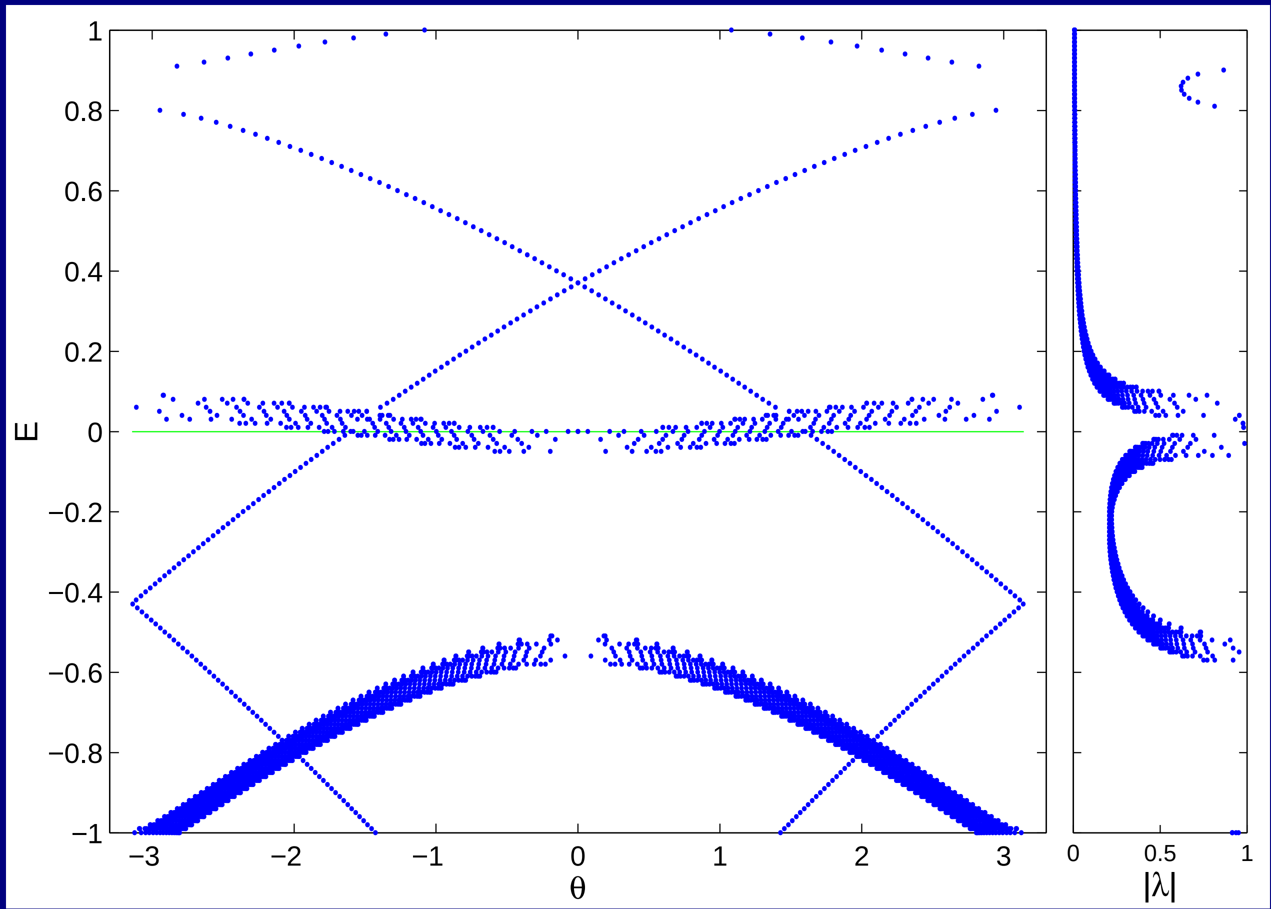
# Example of unstable bands

Dark breather. Cubic potential, attractive coupling $\varepsilon = 0.004$



# Example of stable bands

Dark breather. Cubic potential, repulsive coupling, $\varepsilon = 0.015$



## Degenerate perturbation theory

*If $\mathcal{N}_0$ is a linear operator with a degenerate eigenvalue $E_0$, with eigenvectors $\{|v_n\rangle\}$, which are ortonormal with respect to a scalar product, i.e., $\langle v_n|v_m\rangle = \delta_{n\,m}$, and if $\varepsilon\tilde{\mathcal{N}}$ is a perturbation of $\mathcal{N}_0$, with $\varepsilon$ small; then, to first order in $\varepsilon$, the eigenvalues of $\mathcal{N}_0 + \varepsilon\tilde{\mathcal{N}}$ are $E_0 + \varepsilon\ \lambda_i$, with $\lambda_i$ being the eigenvalues of the perturbation matrix $Q$ with elements $Q_{n\,m} = \langle v_n|\tilde{\mathcal{N}}|v_m\rangle$.*

**Scalar product** $\langle \xi_1|\xi_2\rangle = \sum_{n=1}^{N} \int_{-T/2}^{T/2} \xi_1^*(t)\xi_2(t)\,\mathrm{d}t$

**Basis $N-p$** elements (excited oscillators):

$$|n\rangle = \frac{1}{\mu}\begin{bmatrix} \vdots \\ 0 \\ \dot{u}^0 \\ 0 \\ \vdots \end{bmatrix} \quad ; \quad \mu = \sqrt{\int_{-T/2}^{T/2} (\dot{u}^0)^2\,\mathrm{d}t}$$

**Operators**

$$\mathcal{N}_\varepsilon(u)\,|\xi\rangle = \overbrace{|\ddot{\xi}\rangle + V''(u)\cdot|\xi\rangle}^{\mathcal{N}_0\,|\xi\rangle} + \varepsilon\,\overbrace{\Big(V'''(u)\cdot u_\varepsilon\cdot|\xi\rangle + C\,|\xi\rangle\Big)}^{\tilde{\mathcal{N}}\,|\xi\rangle} =$$
$$= (\underbrace{E_0}_{0} + \varepsilon\,\lambda_i)|\xi\rangle$$

## Demonstration (1)

Deriving with respect to $\varepsilon$, at $\varepsilon = 0$, the dynamical equations we obtain:

$$|\ddot{u}_\varepsilon\rangle + V''(u)\cdot|u_\varepsilon\rangle + C\,|u\rangle = 0 \quad \text{or} \quad N_0|u_\varepsilon\rangle = -C\,|u\rangle\,. \tag{1}$$

$\tilde{C}_{n\,m} = \langle n|C|m\rangle$ is $C$ without the columns and rows corresponding to the oscillators at rest.

$$\langle n|V'''(u)\cdot u_\varepsilon|m\rangle =$$
$$\frac{1}{\mu^2}\int_{-T/2}^{T/2}[\dots,0,\dot{u}_n^0,0,\dots] \quad [\dots,0,V'''(u^0)\,u_{m,\varepsilon}\,\dot{u}_m^0,0,\dots]^\dagger\,\mathrm{d}t =$$
$$\frac{\delta_{n\,m}}{\mu^2}\int_{-T/2}^{T/2}\dot{u}^0\,V'''(u^0)\,u_{n,\varepsilon}\,\dot{u}^0\,\mathrm{d}t\,, \tag{2}$$

with $u_{n,\varepsilon} = \left(\frac{\partial u_n}{\partial\varepsilon}\right)_{\varepsilon=0}$. Thus, $\langle n|V'''(u)\cdot u_\varepsilon\cdot|n\rangle = 0$ if $n \neq m$

To calculate the last integral in (2) we will integrate by parts and use that the integral in a period of the derivative of a periodic function is zero. Besides, the functions $u_{n,\varepsilon}$ are periodic as the coefficients of their Fourier series are given by the derivatives with respect to $\varepsilon$ of the Fourier coefficients of $u_n$. In the deduction below, all the integral limits are $-T/2$ and $T/2$, and the terms between brackets from integration by parts will be zero.

## Demonstration (2)

The last integral in Eq. (2) becomes:

$$
\begin{aligned}
\left[\dot u^0\, u_{n,\varepsilon}\, V''(u^0)\right]_{-T/2}^{T/2} - \int V''(u^0)\dot u^0\, \dot u_{n,\varepsilon}\,\mathrm{d}t - \int V''(u^0)\, \ddot u^0\, u_{n,\varepsilon}\,\mathrm{d}t \ &= \\
-\left[V'(u^0)\, \dot u_{n,\varepsilon},\right]_{-T/2}^{T/2} + \int V'(u^0)\, \ddot u_{\varepsilon,n},\mathrm{d}t - \int V''(u^0)\, \ddot u^0\, u_{n,\varepsilon}\,\mathrm{d}t \ &= \\
-\int \ddot u^0\,(\ddot u_{n,\varepsilon} + V''(u^0)\, u_{n,\varepsilon})\mathrm{d}t \qquad (3)
\end{aligned}
$$

The term between parentheses, is the $n$ component of the lhs of Eq. (1), i.e., it becomes $-\sum_m C_{n\,m}\, u^0_m$, where $u^0_m = u^0$, if the oscillator $m$ is excited, and zero otherwise, i.e., it is $-\sum_m \tilde C_{n\,m}\, u^0 = -(\sum_m \tilde C_{n\,m})\, u^0$. Equation (3) becomes:

$$
\begin{aligned}
(\sum_m \tilde C_{n\,m}) \int \ddot u^0\, u^0\, dt \ &= \\
(\sum_m \tilde C_{n\,m}) \left( \left[\dot u^0\, u^0\right]_{-T/2}^{T/2} - \int (\dot u^0)^2\, dt \right) \ &= \ -(\sum_m \tilde C_{n\,m})\mu^2\,. \qquad (4)
\end{aligned}
$$

That is, equation (2), leads to:

$$
\langle n | V'''(u) \cdot u_\varepsilon \cdot | n \rangle = -\sum_m \tilde C_{n\,m} \quad \text{at} \quad \varepsilon = 0\,. \qquad (5)
$$

## Demonstration (3)

Therefore the diagonal elements of the perturbation matrix $Q$ are

$$Q_{n\,n} = -\sum_{m} \tilde{C}_{n\,m} + \tilde{C}_{n\,n} = -\sum_{\forall\, m\neq n} \tilde{C}_{n\,m} = -\sum_{\forall\, m\neq n} Q_{n\,m} \, . \tag{6}$$

To summarize, the perturbation matrix $Q$ is given by:

$$Q_{n\,m} = \tilde{C}_{n\,m}, \quad n \neq m \quad , \quad Q_{n\,n} = -\sum_{\forall\, m\neq n} Q_{n\,m} \, , \tag{7}$$

$\tilde{C}$ being the coupling matrix without the $p$ rows and columns corresponding to oscillators at rest.

## Perturbation matrix. Oscillators in phase

**Modified coupling matrix $\tilde{C}$** Identical to $C$ but without the rows and columns for the rest oscillators

**Perturbation matrix $Q$**
$$\begin{aligned} Q_{nm} &= \tilde{C}_{nm} \quad , \quad n \neq m \\ Q_{nn} &= -\sum_{\forall m \neq n} Q_{nm} \end{aligned}$$

**Example** : 3–site breather. Code $[1,1,1]$. Elastic attractive interaction

$$C = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} ; \tilde{C} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} ; Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$

$\lambda_1 = 0$: phase mode

$\lambda_2$, $\lambda_3$ positive

**Conclusion** If $\varepsilon > 0$ Unstable for $V$ soft
Stable for $V$ hard

## Perturbation matrix. Oscillators not in phase. $V$ symmetric

**Code matrix** $\sigma = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{bmatrix}$

**Perturbation matrix** $Q$

$$Q_{n\,m} = \left\{ \begin{array}{ccc} \tilde{C}_{n\,m} & \text{if} & \sigma_n = \sigma_m \\ -\tilde{C}_{n\,m} & \text{if} & \sigma_n \neq \sigma_m \end{array} \right\} \quad n \neq m$$

$$Q_{n\,n} = -\sum_{\forall\, m \neq n} Q_{n\,m}$$

## Symmetric multibreathers stability theorem

*Let be $V(u_n)$ symmetric and $\{\lambda_i\}$ the eigenvalues of $Q_{nm}$:*
*(a) if $V(u_n)$ is hard and there is any negative value in $\{\epsilon\lambda_i\}$ the multibreather at low coupling will be unstable, and stable otherwise.*
*(a) if $V(u_n)$ is soft and there is any positive value in $\{\epsilon\lambda_i\}$ the multibreather at low coupling will be unstable, and stable otherwise.*

**Example** : 3–site breather. Code $[-1, 1, -1]$. Elastic attractive coupling.

$$C = \begin{bmatrix} 2 & -1 & 0 \cdots & & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix};\ \tilde{C} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix};\ Q = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0$ (phase mode), $\lambda_2 = -1$, $\lambda_3 = -3$ (both negative)
**Conclusion**: If $\varepsilon > 0$: Stable for $V$ soft, Unstable for $V$ hard.

## Non–symmetric on–site potentials

**Symmetry coefficient** $\gamma$ : $\gamma = -\dfrac{\displaystyle\int_{-T/2}^{T/2} \dot{u}^0(t)\, \dot{u}^0(t+T/2)\,\mathrm{d}t}{\displaystyle\int_{-T/2}^{T/2} \dot{u}^0(t)\, \dot{u}^0(t)\,\mathrm{d}t}$

**Properties of** $\gamma$

1) $\gamma = \gamma(\omega_\mathrm{b})$
2) $0 < \gamma < 1$
3) If $V$ is symmetric $\gamma = 1$
4) $\omega_\mathrm{b} \longrightarrow \omega_0 \quad \Rightarrow \quad \gamma \longrightarrow 1$
5) Numerically: Fourier coefficients
6) Analytically: Morse potential: $\gamma = \omega_\mathrm{b}$

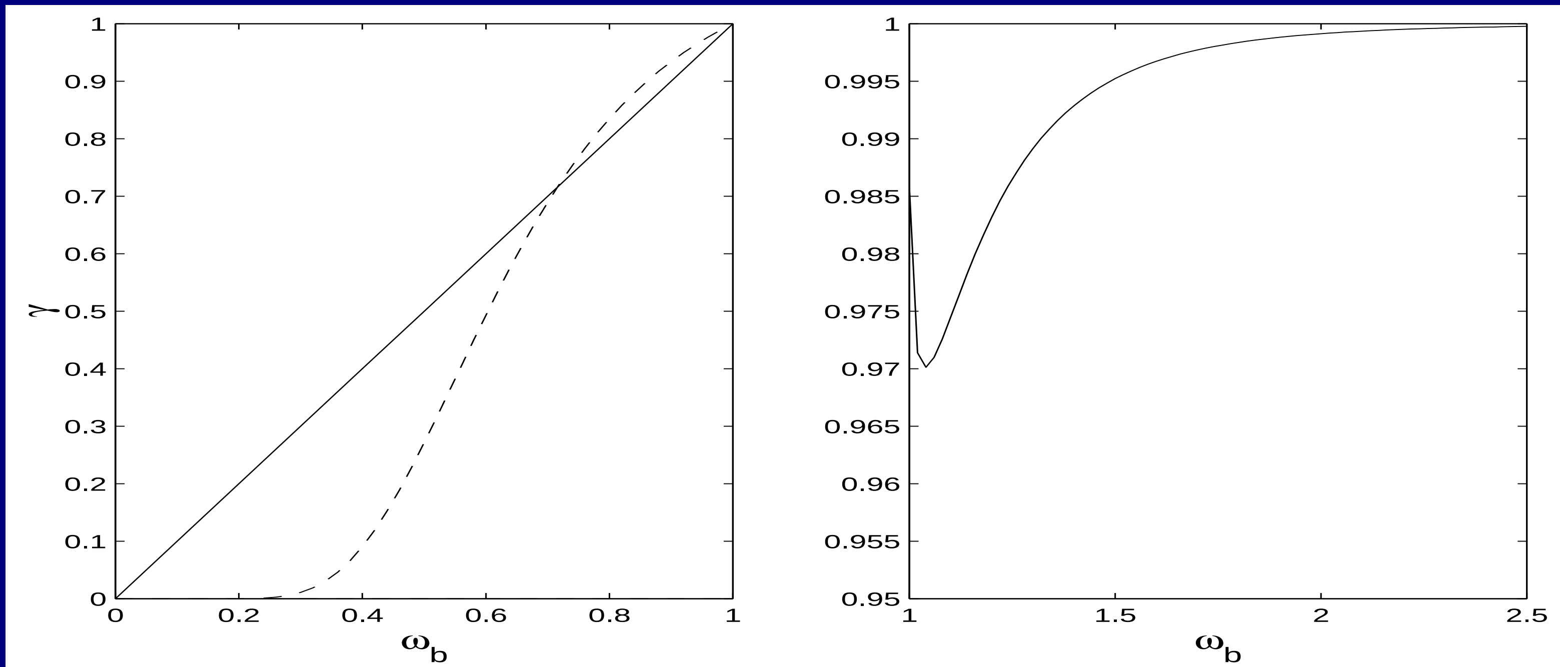
**Perturbation matrix** $Q$

$$Q_{n\,m} = \left\{ \begin{array}{ccc} \tilde{C}_{n\,m} & \text{if} & \sigma_n = \sigma_m \\ -\gamma\,\tilde{C}_{n\,m} & \text{if} & \sigma_n \neq \sigma_m \end{array} \right\} \quad n \neq m$$

$$Q_{n\,n} = -\sum_{\forall\, m \neq n} Q_{n\,m}$$

**Non–symmetric multibreather stability theorem**

## The symmetry coefficient $\gamma$ versus $\omega_{\rm b}$




**Soft** **Hard**

– – – – $V(u_n) = \frac{1}{2}u_n^2 - \frac{1}{3}u_n^3$

——— $V(u_n) = \frac{1}{2}(\exp(-u_n) - 1)^2$

$V(u_n) = \frac{1}{2}u_n^2 + \frac{1}{3}u_n^3 + \frac{1}{4}u_n^4$

Rest frequency $\omega_0 = 1$

## Application. 2-site breathers

$$C = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Code $\pm[1,1]$: $\tilde{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$; $Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; $(\lambda_1, \lambda_2) = (0, +2)$

Code $\pm[-1,1]$: $\tilde{C} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $Q = \begin{bmatrix} -\gamma & \gamma \\ \gamma & -\gamma \end{bmatrix}$; $(\lambda_1, \lambda_2) = (0, -2\gamma)$

| $\pm[1,\ 1]$ | Attractive | Repulsive |
|---|---|---|
| Soft | Unstable | Stable |
| Hard | Stable | Unstable |

| $\pm[1, -1]$ | Attractive | Repulsive |
|---|---|---|
| Soft | Stable | Unstable |
| Hard | Unstable | Stable |

Theorem by Aubry for the symmetric case in:
JL Marín. *PhD thesis*, June 1997.

# Application. 3–site breathers

Code $\pm[1,1,1]$ :

| | Attractive coupling | Repulsive coupling |
|---|---|---|
| Soft | Unstable | Stable |
| Hard | Stable | Unstable |

Code $\pm[-1,1,-1]$ :

| | Attractive coupling | Repulsive coupling |
|---|---|---|
| Soft | Stable | Unstable |
| Hard | Unstable | Stable |

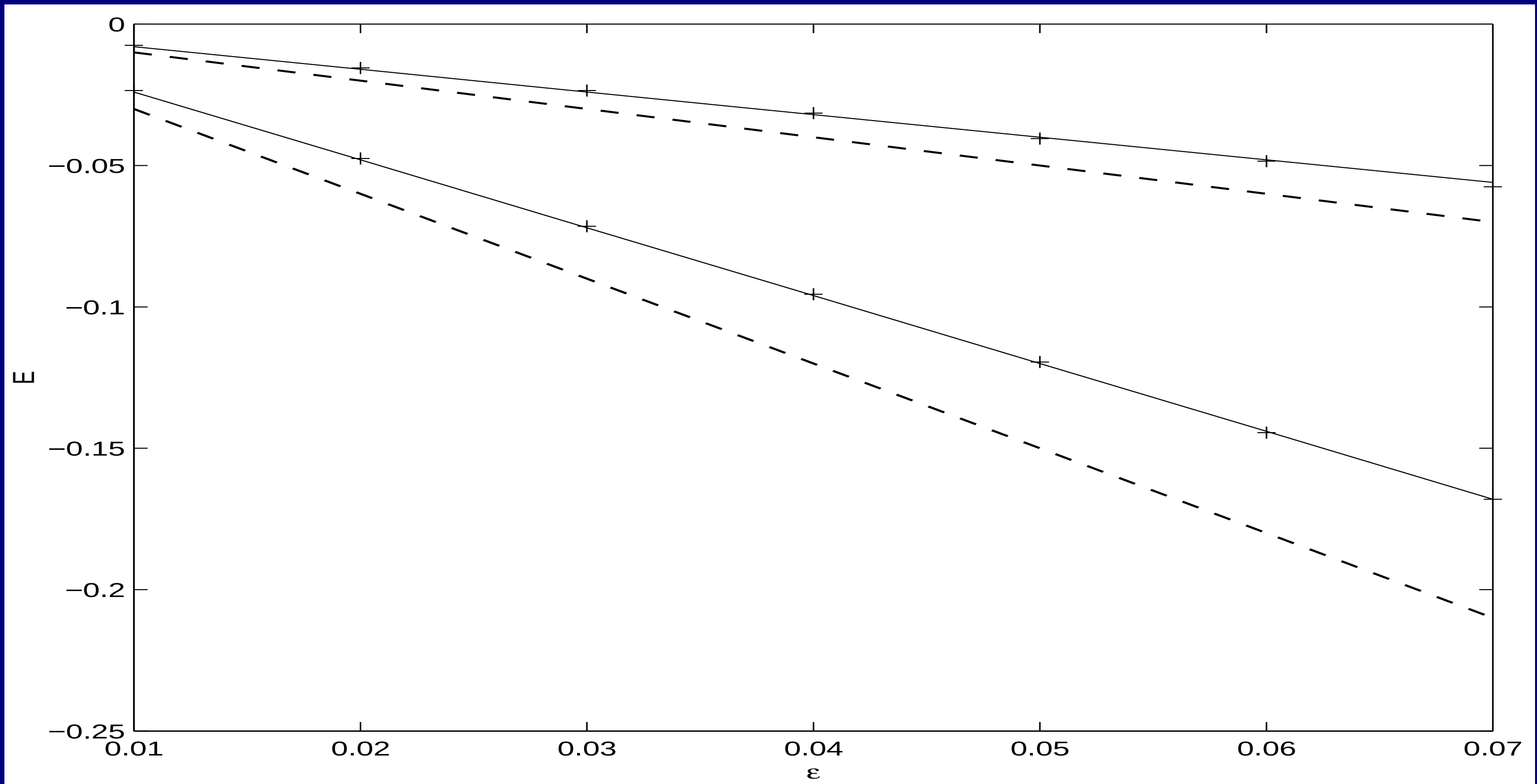
Code $\pm[1,1,-1]$ :

Always unstable

## MST versus *exact* numerical

Morse potential. Elastic attractive coupling. Code $[-1, 1, -1]$.
Frequency $\omega_{\rm b} = \gamma = 0.8$.

Eigenvalues $E$ of the Newton operator versus the coupling $\varepsilon$




$+$ Numerical. $----$ Symmetric MST. ——— Non–symmetric MST

## Multibreathers

Group of contiguous oscillators excited at $\varepsilon = 0$

In–phase (wave vector $q = 0$) :

| | Attractive coupling | Repulsive coupling |
|---|---|---|
| Soft | Unstable | Stable |
| Hard | Stable | Unstable |

Out–of–phase (wave vector $q = \pi$) :

| | Attractive coupling | Repulsive coupling |
|---|---|---|
| Soft | Stable | Unstable |
| Hard | Unstable | Stable |

Mixed : Always unstable

## Phonobreathers

All oscillators excited at zero coupling

- With **free–ends** or **fixed–ends** boundary conditions: as multibreathers

  Coherent with:

  I. Daumont, T. Dauxois, and M. Peyrard.
  *Nonlinearity*, 10:617–630, 1997. (Modulational instability).

  AM Morgante, M Johansson, G Kopidakis, and S Aubry.
  *Phys. D*, 162:53, 2002. (DNLS)

  S Aubry. *Physica D*, 103:201–250, 1997. (Theorem 9, action properties)

- With **periodic** boundary conditions: **Parity instabilities**

## Dark breathers

- **Free–ends** of **fixed–ends**: almost as multibreathers:
  There is a degenerate 0-eigenvalue of $Q$ $\Rightarrow$
  If the eigenvalues of $Q$ correspond to instability: instability
  If the eigenvalues of $Q$ correspond to stability: undefined
- Periodic boundary conditions: **Parity instabilities**
- **Parity instabilities. Example.**
  Dark breather. Attractive coupling. Soft on-site potential.
  Code $\sigma = [\,\cdots, 1, -1, 0, 1, -1, \cdots]$

  $N$ **odd** ($N = 5$):
  $\sigma = [\,1, -1, 0, 1, -1]$: **stable**

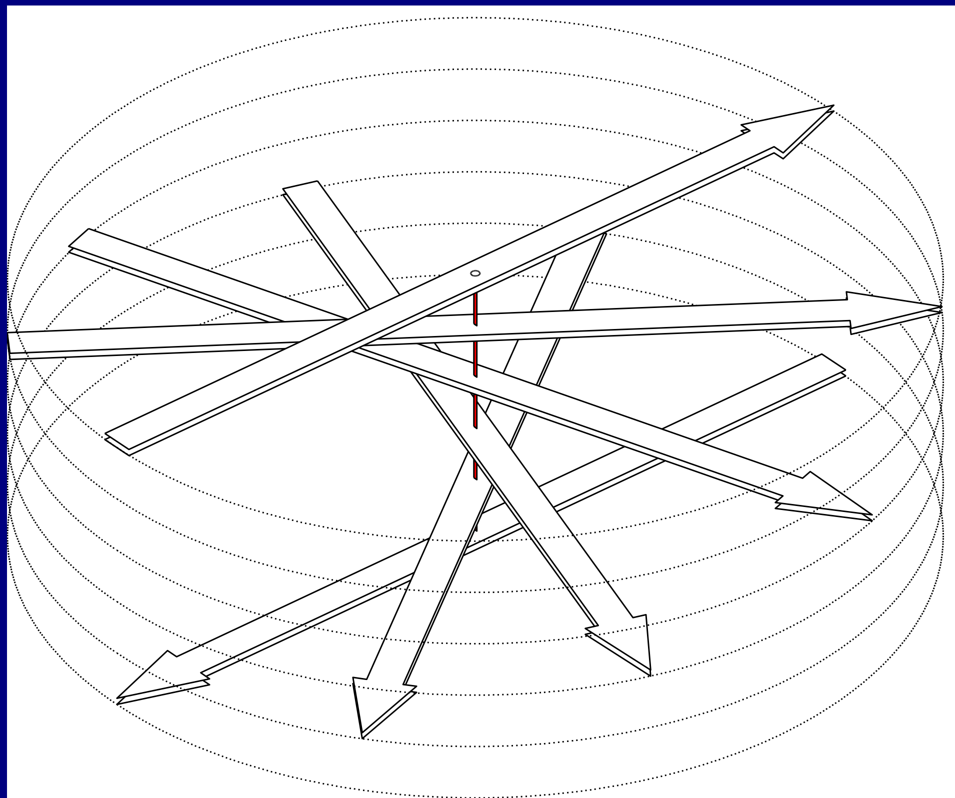
  $N$ **even** ($N = 6$): $\sigma = [\,1, -1, 0, 1, -1, 1]$
  equivalent to $\sigma = [-1, 0, 1, -1, 1, 1]$: **unstable**.

  But with code $\sigma = [\,\cdots, -1, 1, 0, 1, -1, \cdots]$ the conditions are reversed.

# A model with LRI (1)

Twist model for DNA:

B Sánchez-Rey, JFR Archilla, F Palmero, and FR Romero.
*Phys. Rev. E*, 66:017601–017604, 2002.
Selected by Virtual Journal of Biological Physics Research 4(1), 2002.



## A model with LRI (2)

**Dynamical equations**

$$\ddot{u}_n + V'(u_n) + \varepsilon \sum_{m=n-N/2}^{n+N/2} \frac{\cos[\theta_{tw}(n-m)]}{|n-m|^3} u_m = 0$$

$V(u_n)$: Morse potential

**Coupling matrix** $C_{n\,m} = \dfrac{\cos[\theta_{tw}(n-m)]}{|n-m|^3}$

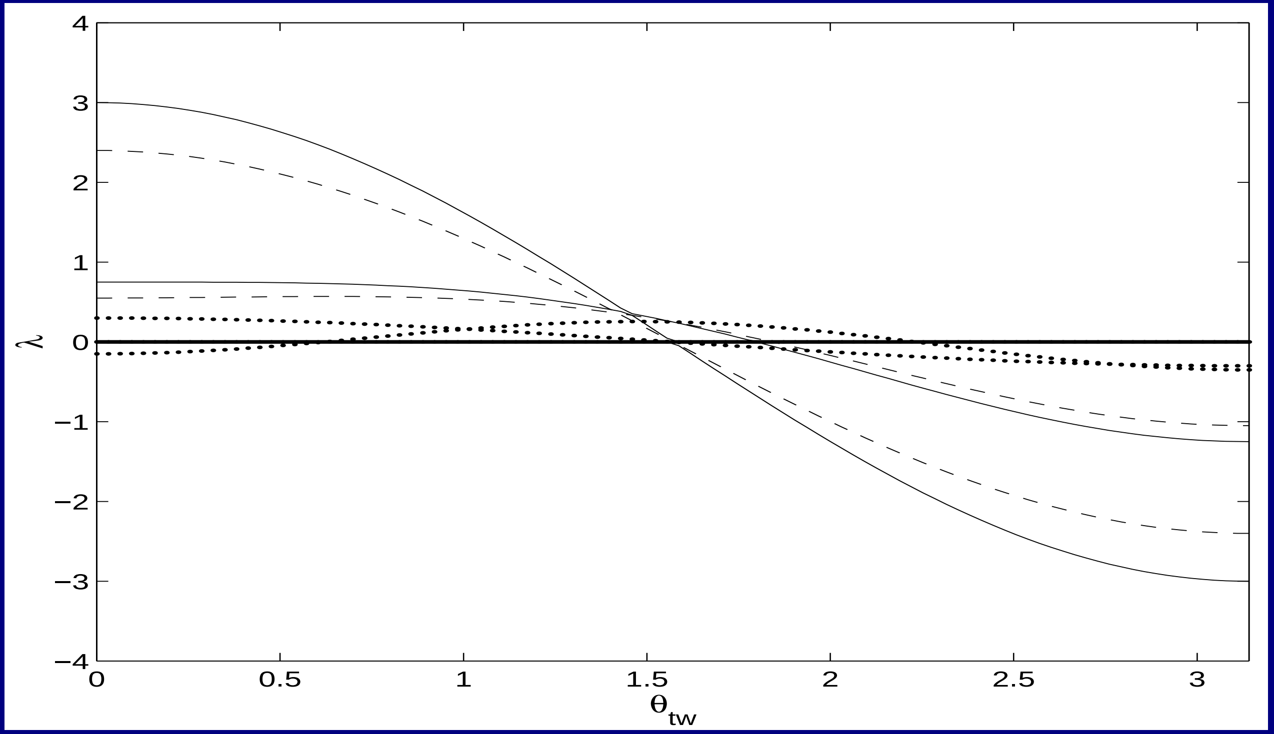
**Some results**

| Code | $\theta_{tw} = 0$ | $\theta_{tw} = \pi$ |
|---|---|---|
| 11 | Stable | Unstable |
| 1 -1 | Unstable | Stable |
| 101 | Stable | Stable |
| 111 | Stable | Unstable |
| 1-1 1 | Unstable | Stable |
| 11-1 | Unstable | Stable |

# A model with LRI (3)

Dependence on the symmetry coefficient $\gamma = \omega_{\rm b}$. Code $[-1, 1, -1]$.

Eigenvalues $\lambda_i$ of $Q$ versus the twist angle $\theta_{tw}$




——— : $\gamma = \omega_{\rm b} = 1$ - - - - : $\gamma = \omega_{\rm b} = 0.8$ $\cdot\ \cdot\ \cdot\ \cdot$ : $\gamma = \omega_{\rm b} = 0.1$

Instability induced by the nonlinearity for $\gamma = 0.1$ and $\theta_{tw} = 2\,$rad.

# Multibreathers in the Peyrard–Bishop DNA model

**Dynamical equations** Planar model

$$\ddot{u}_n + V'(u_n) + \epsilon\,(2\,u_n - u_{n-1} - u_{n+1}) = 0$$

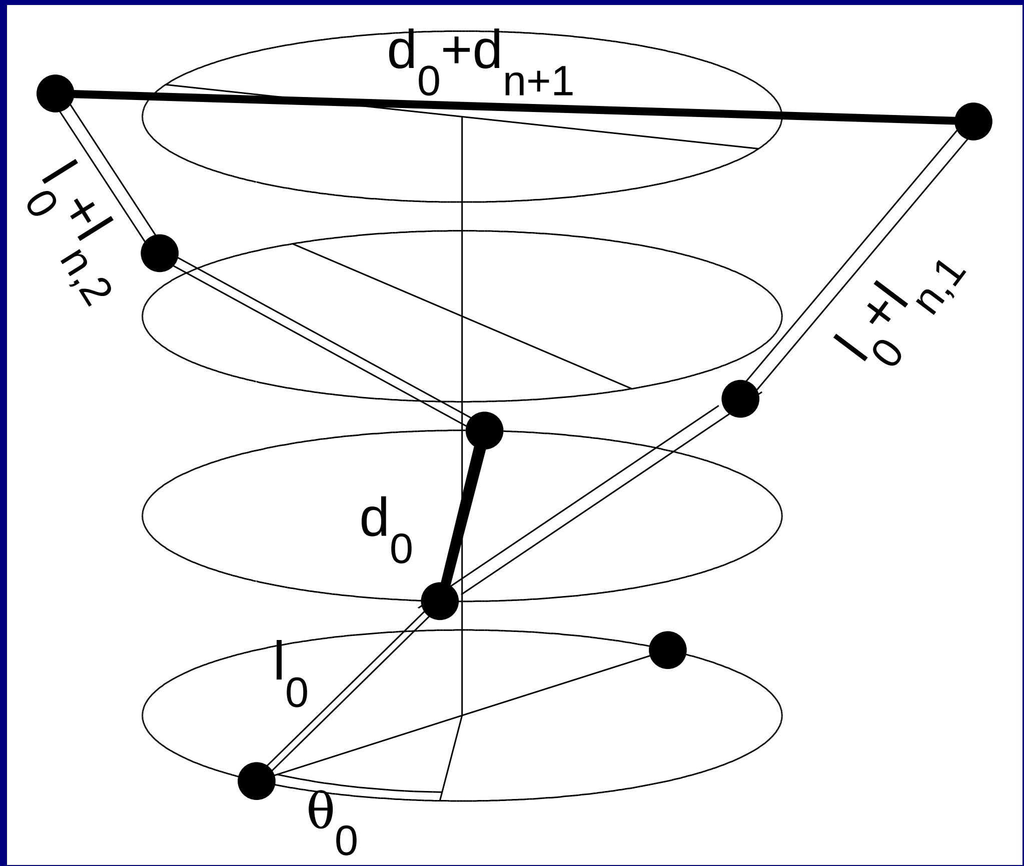
$V$: Morse potential (soft) and attractive coupling

In-phase multibreathers are unstable

**Helical model for DNA**

- D Hennig and JFR Archilla, *Multi-site H-bridge breathers in a DNA–shaped double strand.* Physica Scripta, 69(2):150-160, 2004.
- Variant of:
  M. Barbi, S. Cocco and M. Peyrard, *Phys. Lett. A* **253**, 358 (1999). And continuations.

# A helical DNA–like model



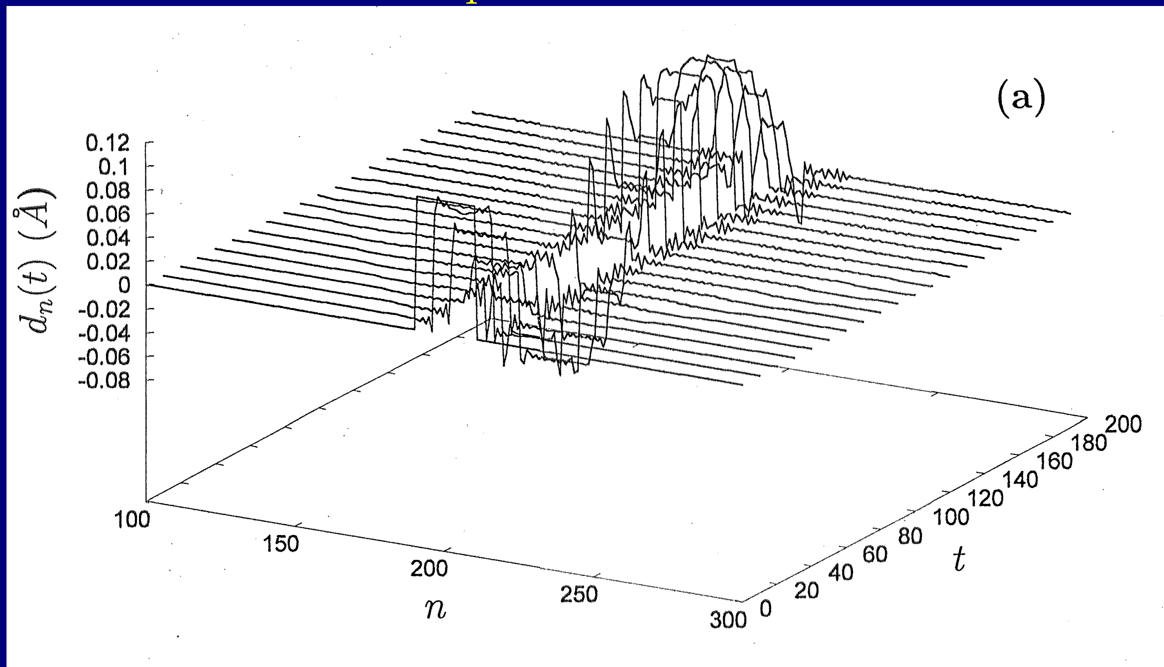
# Multibreathers in the helical DNA–like model

**Effective repulsive coupling**

**Linear equations**

$$\ddot{r}_n = -\omega_0^2\, r_n - \varepsilon\, \sin^2(\frac{\theta_0}{2})\,(2\, r_n + r_{n+1} + r_{n-1}) - \varepsilon\, \sin(\frac{\theta_0}{2})\cos(\frac{\theta_0}{2})\, d_0\,(\alpha_{n+1} - \alpha_{n-1})\,,$$

Stable in–phase multibreathers




**The helical shape supports in–phase, stable multibreathers**

## Generalization

**Hamiltonian** $H = \sum_n \left(\frac{1}{2}\, m_n\, \dot{u}_n^2 + V_n(u_n)\right) + \varepsilon\, W(u)$

**Dynamical equations** $m_n\, \ddot{u}_n + V_n'(u_n) + \varepsilon \dfrac{\partial W(u)}{\partial u_n} = 0$

$V_n(u_n)$ heterogeneous
multiple wells, etc

**Given periodic solution at** $\varepsilon = 0$ **:** $u^0 = \begin{bmatrix} u_1^0 \\ \vdots \\ u_N^0 \end{bmatrix}$

$u_n^0$ determined by
$n$, well, phase
(non time–reversible)

**Basis elements :** $|n\rangle = \dfrac{1}{\mu_n} \begin{bmatrix} \vdots \\ 0 \\ \dot{u}_n^0 \\ 0 \\ \vdots \end{bmatrix}$ ; $\mu_n = \sqrt{\int_{-T/2}^{T/2} (\dot{u}_n^0)^2 \mathrm{d}t}$

**Perturbation matrix**

$$Q_{n\,m} = \frac{1}{\mu_n\,\mu_m} \int_{-T/2}^{T/2} \dot{u}_n^0\, \frac{\partial^2 W(u^0)}{\partial u_n\, \partial u_m}\, \dot{u}_m^0\, \mathrm{d}\,t\,, \quad n \neq m\,; \quad Q_{n\,n} = -\sum_{\forall\, m \neq n} \frac{\mu_m}{\mu_n}\, Q_{n\,m}$$

**Generalized multibreathers stability theorem**

## Conclusions

**1** A theory for calculating the stability of **any** multibreather in **any** Klein–Gordon system at (relatively) low coupling

**2** For (relatively) simple systems is very simple

**3** For complex systems is (relatively) complex

**4** Application to a number of systems

| | |
|---|---|
| 2–site breathers | 3–site breathers |
| Multibreathers | Phonobreathers |
| Dark breathers | Parity instabilities |
| Systems with LRI | Nonlinear instabilities |

**5** Potential consequences for helical DNA models

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