Some quantum lattice problems

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Summary:

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  - DNLS lattice
- Fermi resonance lattice
  - 1 site case
  - 2 site case
  - f site case
- Quasi-exactly solvable model
  - arbitrary cutoff m, monomer case
  - arbitrary cutoff m, dimer case
  - $m = 2$, arbitrary f

Introduction

Quantum DNLS (boson Hubbard) Hamiltonian in 1D, nearest neighbour interactions:

$$\hat{H} = -\frac{\gamma}{2}\sum_{j=1}^{f} b_j^\dagger b_j^\dagger b_j b_j - \sum_j b_j^\dagger (b_{j+1} + b_{j-1})$$

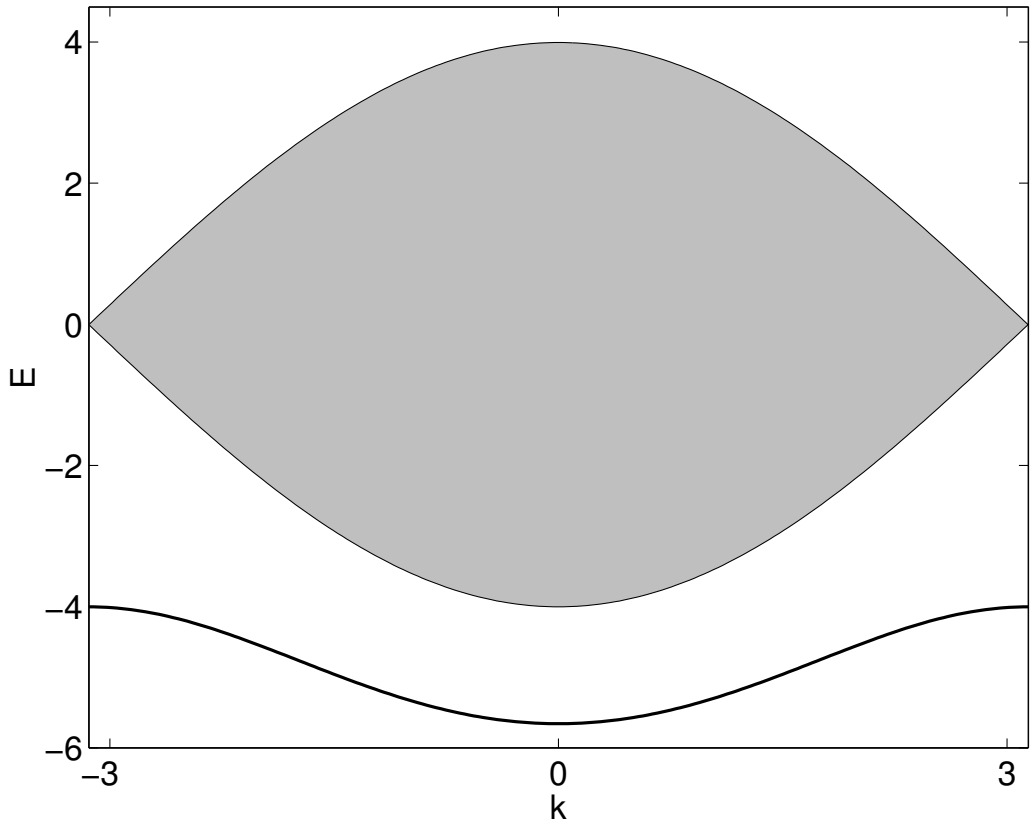
$\hat{H}$ conserves the *number* of quanta

$$\hat{N} = \sum_{j=1}^{f} b_j^\dagger b_j \ ,$$

The operators b_j, $(b_j^\dagger)$ destroy (create) a boson at site j on the lattice of size f. We assume periodic boundary conditions.

1D "Quantum Breather"

Simplest non-trivial case, lattice with only 2 bosons.



Eigenvalues $E(k)$ for QDNLS. The lower band is the "breather" band. Similar results in 2- and D- dimensions.

Modified QDNLS

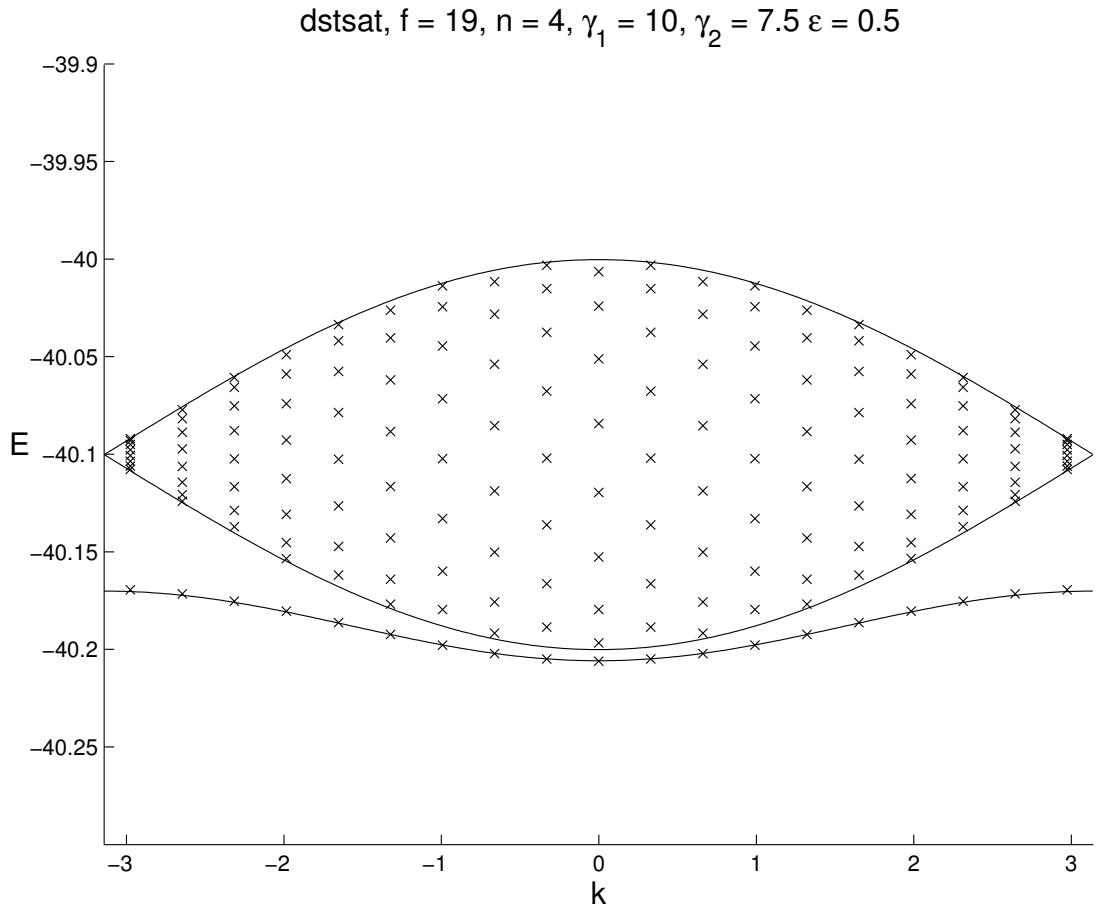
A disadvantage of the QDNLS is that the energy of the ground state $\Rightarrow -\infty$ as the number of bosons increases. To avoid this we can use the modified QDNLS Hamiltonian:

$$\hat{H} = -\frac{1}{2}\gamma_1 \sum_{j=1}^{f} b_j^\dagger b_j^\dagger b_j b_j + \frac{1}{2}\gamma_2 \sum_{j=1}^{f} b_j^\dagger b_j^\dagger b_j^\dagger b_j b_j b_j - \\ - \sum_j \left(b_j^\dagger b_{j+1} + b_j^\dagger b_{j-1} \right)$$

This is equivalent to adding a saturation term in nonlinear optics.

Modified QDNLS

We can now get some interesting detailed band structure in the 4-boson case which is reminiscent of the 2-boson case



see Dorignac et al., PRL **93** 025504, 2004.

Fermi Resonance Model

Now *two* types of bosons, a and b

$$H = \sum_{n=1}^{f} \omega_a a_n^\dagger a_n + \omega_b b_n^\dagger b_n - \gamma_a a_n^\dagger a_n^\dagger a_n a_n$$
$$- \gamma_b b_n^\dagger b_n^\dagger b_n b_n - (b_n^\dagger b_n^\dagger a_n + a_n^\dagger b_n b_n)$$
$$- \eta a_n^\dagger (a_{n+1} + a_{n-1}) - \chi b_n^\dagger (b_{n+1} + b_{n-1})$$

The Hamiltonian commutes with the number operator

$$\hat{N} = \sum_{n=1}^{f} 2a_n^\dagger a_n + b_n^\dagger b_n$$

We will use m for the number of a bosons and ℓ for the number of b bosons, so

$$N = 2m + \ell$$

is conserved.

The monomer case

We write the wave function of a monomer with m type a bosons and ℓ type b bosons as

$$|{}^{m}_{\ell}\rangle \equiv [{}^{m}_{\ell}]$$

For each $N = 2m + \ell$, we have $M + 1$ different states, where M is the integer part of $N/2$

$$\left[{}^{M}_{N-2M}\right], \left[{}^{M-1}_{N-2(M-1)}\right], \ldots, \left[{}^{0}_{N}\right]$$

for even N, we have the tridiagonal $(N/2+1) \times (N/2+1)$ Hamiltonian

$$H = \begin{bmatrix} d_1 & c_1 & 0 & & & \\ c_1 & d_2 & c_2 & 0 & & \\ 0 & c_2 & d_3 & c_3 & 0 & \\ & 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where

$$d_i = -(2i-2)(2i-3)\gamma_b + (M-i+1)\omega_a + \\ + 2(i-1)\omega_b - (M-i)(M-i+1)\gamma_a,$$

$$c_i = -\sqrt{2i(2i-1)(M-i+1)}.$$

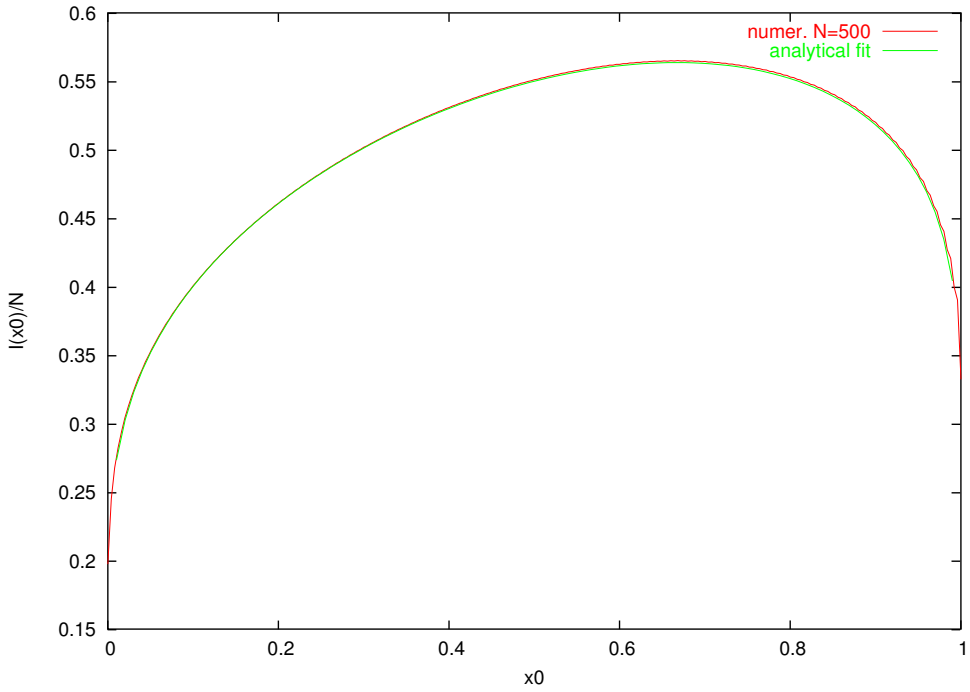
Monomer case – applications

The eigenvectors in the monomer case can be calculated numerically in a straightforward way. However the dynamics of the monomer case have an interesting application - the atomic–molecular Bose-Einstein condenstate. Now a is the atomic field and b the molecular field, and the model is simplified

$$H = \frac{1}{2}\Delta a_n^\dagger a_n + \frac{\chi}{2\sqrt{V}}(b_n^\dagger b_n^\dagger a_n + a_n^\dagger b_n b_n).$$

We can integrate the time-dependent problem numerically using the eigenvalues and eigenvectors of H, but Dorignac and Gaididei et al. have used the discrete WKB approach to find some nice analytic approximations, much more accurate than the normal mean-field approximation.

Monomer case – applications

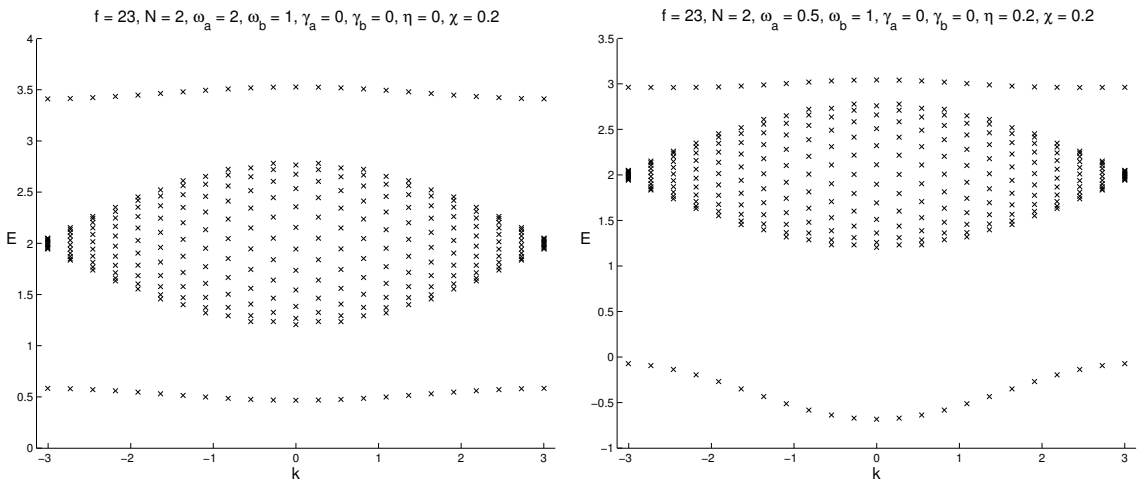


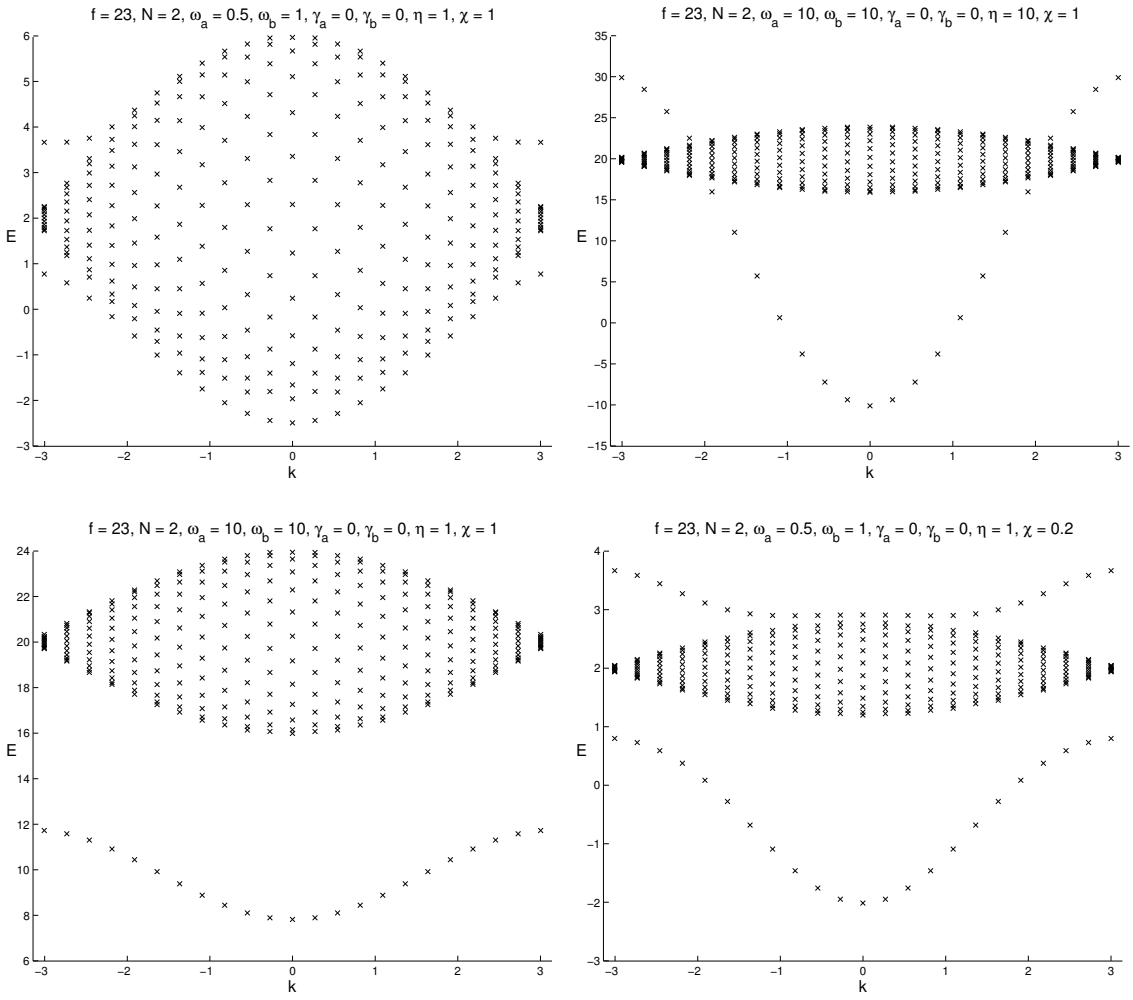
The figure shows the comparison between the "exact" numerical and the approximate values of the dc-component of the fraction of atomic states as a function of the initial number of atoms in the system.

$N = 2$, general f

We can calculate results for the dimer for small N, but if we restrict ourselves to the case $N = 2$ it is possible to get results for general f.

This corresponds to a generalization of the $n = 2$ case for the QDNLS model. We get again a tridiagonal Hamiltonian matrix, giving various band structures even when $\gamma_a = \gamma_b = 0$, i.e. the nonlinearity comes from the interaction term alone.





Caisar Katerji has also done a careful study of the dynamics in various cases.

Quasi-exactly solvable model

A major feature to the QDNLS family of models is that there is a *conserved* number of bosons. In seeking to understand models with no conserved number, we have been studying the following model

$$\begin{aligned}\hat{H} = &-\gamma_1 \sum_{j=1}^{f} b_j^\dagger b_j^\dagger b_j b_j + \gamma_2 \sum_{j=1}^{f} b_j^\dagger b_j^\dagger b_j^\dagger b_j b_j b_j + \\ &- \epsilon \sum_{j} \left(b_j^\dagger b_{j+1} + b_j^\dagger b_{j-1} \right) \\ &+ \beta \sum_{j=1}^{f} \left[b_j^\dagger \left(\hat{N} - m \right) + \left(\hat{N} - m \right) b_j \right]\end{aligned}$$

The first three terms are as before, and conserve N, but the final term does not, *except* when $N = m$.

Quasi-exactly solvable model

The result of the m-dependent term is to divide the Hamiltonian matrix into two blocks, a finite size one and an infinite one

$$H = \begin{bmatrix} H^m & 0 \\ 0 & H^\infty \end{bmatrix}.$$

By adjusting the value of γ_1 and γ_2 we can make sure that some of the eigenvalues of H^m lie below all those of H^∞.

In the *monomer* case H^m, is tridiagonal with diagonal elements

$$H^m_{n,n} = -\gamma_1\, n\,(n-1) + \gamma_2\, n\,(n-1)\,(n-2)$$

and off-diagonal

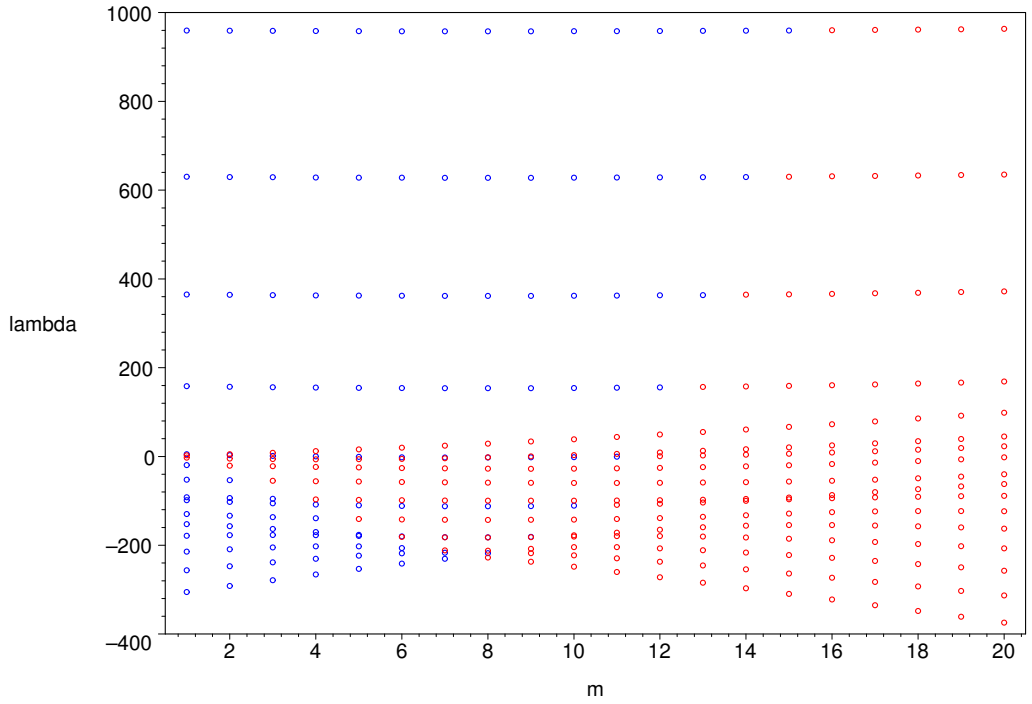
$$H^m_{n,n+1} = H^m_{n+1,n} = -\beta\,(m-n)\sqrt{n+1}$$

Quasi-exactly solvable model - dimer case

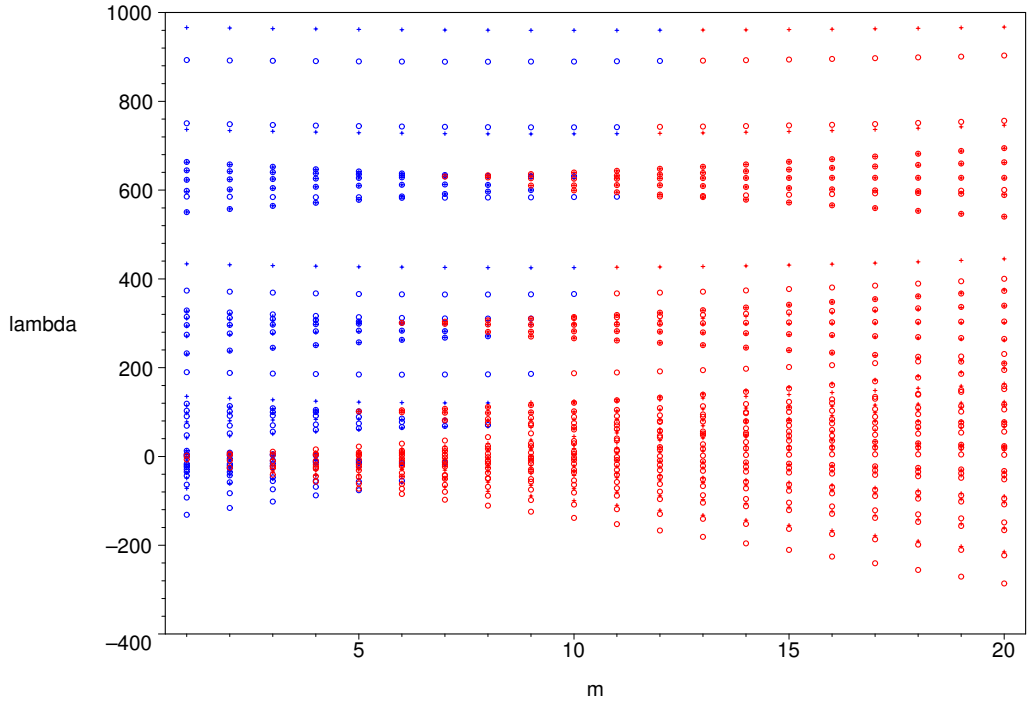
We can split the basis states into symmetric ($k = 0$) and anti-symmetric ($k = 1$) states, then H^m and H^∞ split also. Now $H^{(m,k)}$ is *block*-tridiagonal.

$$H^{(k,m)} = \begin{bmatrix} \ddots & \ddots & & \bigcirc \\ \ddots & H^{(k,m)}_{n,n} & H^{(k,m)}_{n,n+1} & \\ & H^{(k,m)}_{n+1,n} & \ddots & \ddots \\ \bigcirc & & \ddots & H^{(k,m)}_{m,m} \end{bmatrix}.$$

and the formula for the elements of each block can be written down.



monomer case, infinite matrix, finite matrix,
$\gamma_1 = 10$, $\gamma_2 = 1$, $\beta = 2.5$



dimer case, $\circ$ symm., $+$ anti-symm., $\epsilon = 2.5$.

Quasi-exactly solvable model
$m = 2$, general f case

In this case we can also write down exact equations for the *block*-tridiagonal matrix H^m (see Brihaye, `quant-ph/0412174`).