SOME THEOREMS ON THE STABILITY OF MULTIBREATHERS IN KLEIN-GORDON LATTICES

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Klein–Gordon systems with linear coupling

Dynamical equations

$$\ddot{u}_n + V'(u_n) + \varepsilon \sum_{m=1}^{N} C_{n\,m} u_m = 0 \quad n = 1, \dots, N$$

Coupling examples ($\varepsilon > 0$):

- Elastic attractive coupling: $\sum_{m=1}^{N} C_{n\,m} u_m = 2\,u_n - u_{n+1} - u_{n-1}$
- Next–neighbor, dipole–dipole repulsive coupling: $\sum_{m=1}^{N} C_{n\,m} u_m = u_{n+1} + u_{n-1}$
- Dipolar long–range interaction repulsive coupling: $\sum_{m=1}^{N} C_{n\,m} u_m = \sum_{m=1}^{N} \frac{u_m}{r_{n\,m}^3}$

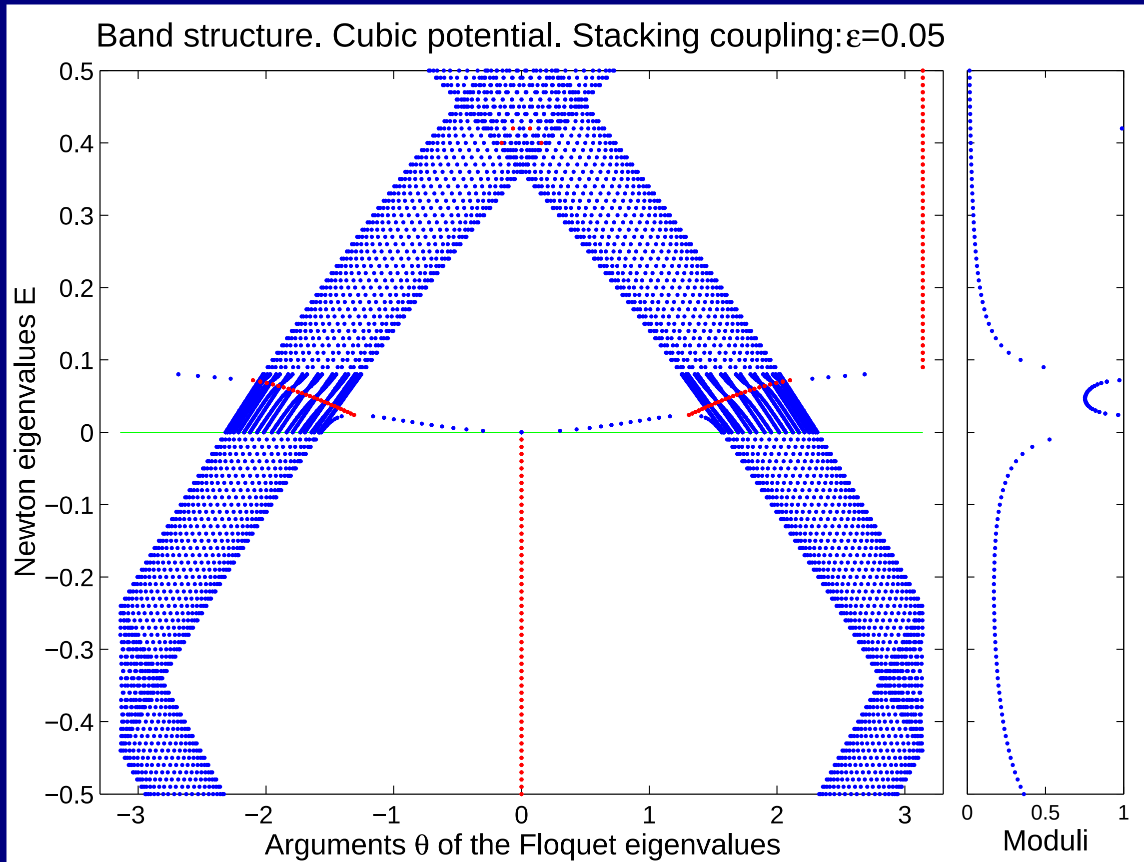
Notation $|u\rangle = \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t)] \end{bmatrix}$; $|V(u)\rangle = \begin{bmatrix} V(u_1) \\ \vdots \\ V(u_N) \end{bmatrix}$; $|\dot{u}\rangle = \begin{bmatrix} \frac{\mathrm{d}u}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}u_N}{\mathrm{d}t} \end{bmatrix}$; $|V'(u)\rangle = \begin{bmatrix} \frac{\partial V}{\partial u_1} \\ \vdots \\ \frac{\partial V}{\partial u_N} \end{bmatrix}$

Linear stability and Newton operator

- Dynamical equation: $|\ddot{u}\rangle \,+\, |V'(u)\rangle \,+\, \varepsilon\, C\, |u\rangle \,=\, 0$
  $|u\rangle$ time–reversible and periodic with frequency $\omega_{\rm b} = \dfrac{2\pi}{T}$
- (Linear) stability equation
$$\mathcal{N}_\varepsilon(u)\,|\xi\rangle \;\equiv\; |\ddot{\xi}\rangle \,+\, V''(u)\cdot|\xi\rangle \,+\, \varepsilon\, C\,|\xi\rangle =\; E\,|\xi\rangle$$
  Newton operator: $\mathcal{N}_\varepsilon$
- The Floquet Matrix: $\left[\begin{array}{c}\{\xi_n(T)\}\\ \{\dot{\xi}_n(T)\}\end{array}\right] = \mathcal{F}_E \left[\begin{array}{c}\{\xi_n(0)\}\\ \{\dot{\xi}_n(0)\}\end{array}\right]$
- Floquet multipliers of $\mathcal{F}_E$ and arguments:
$$\lambda_i = \exp({\rm i}\theta_i) \quad ; \quad i = 1,\ldots,2\,N$$
- Band structure
$$(\{\theta_l\}, E) \quad \text{with} \quad \theta_l \in \mathcal{R}$$
- Stability:
  The solution $|u\rangle$ is linearly stable if there are 2N band intersections or tangent points with their multiplicity with the axis $E = 0$.
  S Aubry. *Physica D*, 103:201–250, 1997.

Example of bands

Cubic potential, attractive elastic coupling, $\varepsilon = 0.05$.
Stable single breather.



Bands figures from:
A Alvarez, JFR Archilla, J Cuevas and FR Romero, *Dark breathers in Klein-Gordon lattices. Band analysis of their stability properties.* New Journal of Physics 4:72.1-72.19 (2002).

Bands at the anticontinuous limit ($\varepsilon = 0$)

Stability equation: $\mathcal{N}_0(u)\,|\xi\rangle \;\equiv\; |\ddot{\xi}\rangle \,+\, V''(u)\cdot|\xi\rangle \;=\; E\,|\xi\rangle\,,$

- $N-p$ excited oscillators . Code $\sigma_n = \pm 1$

$$\ddot{\xi}_n \,+\, V''(u_n)\,\xi_n \,=\, E\,\xi_n$$

  With $E=0$:
  Phase mode (periodic): $\dot{u}_n$
  Growth mode (unbounded): $\dfrac{\partial u_n}{\partial \omega_\mathrm{b}}$

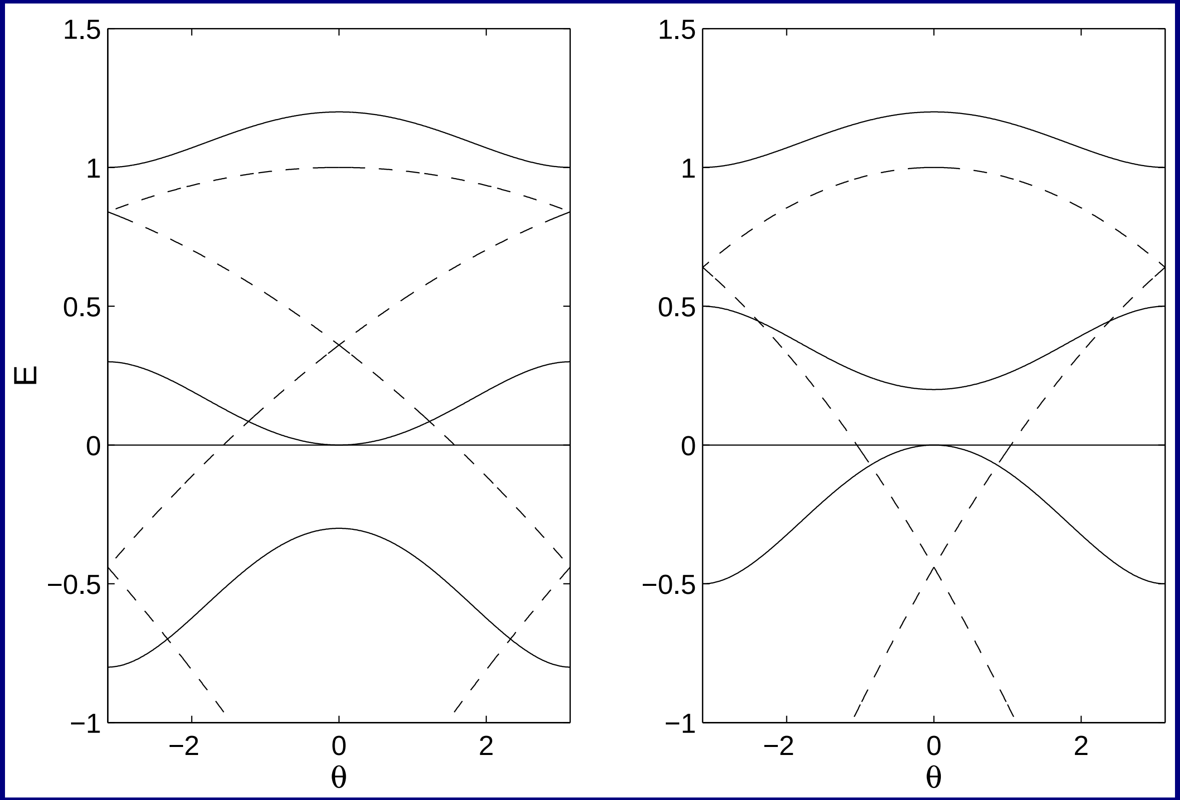
- p rest oscillators. Code $\sigma_n = 0$

$$\ddot{\xi}_n \,+\, (\omega_0)^2 \xi_n \,=\, E\,\xi_n \quad ; \quad \omega_0 = \sqrt{V''(0)}$$

Bands

- Excited oscillators ($N-p$): Tangent from above (soft) or below (hard)
- Rest oscillators (p): $E = \omega_0^2 - \omega_\mathrm{b}^2 (\dfrac{\theta}{2\pi})^2$

Bands at the anticontinuous limit

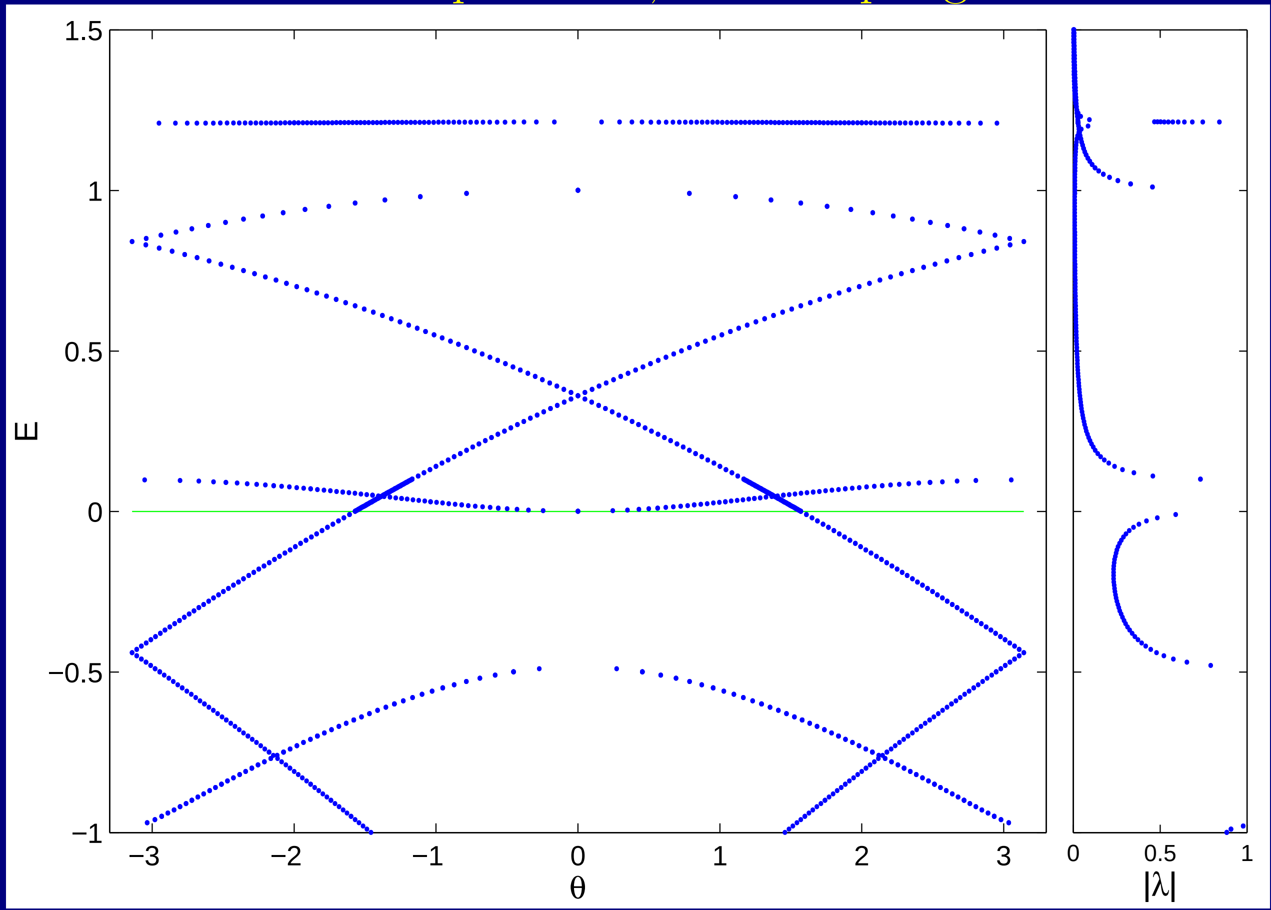


Soft Hard

- - - - Rest oscillators ——— Excited oscillators

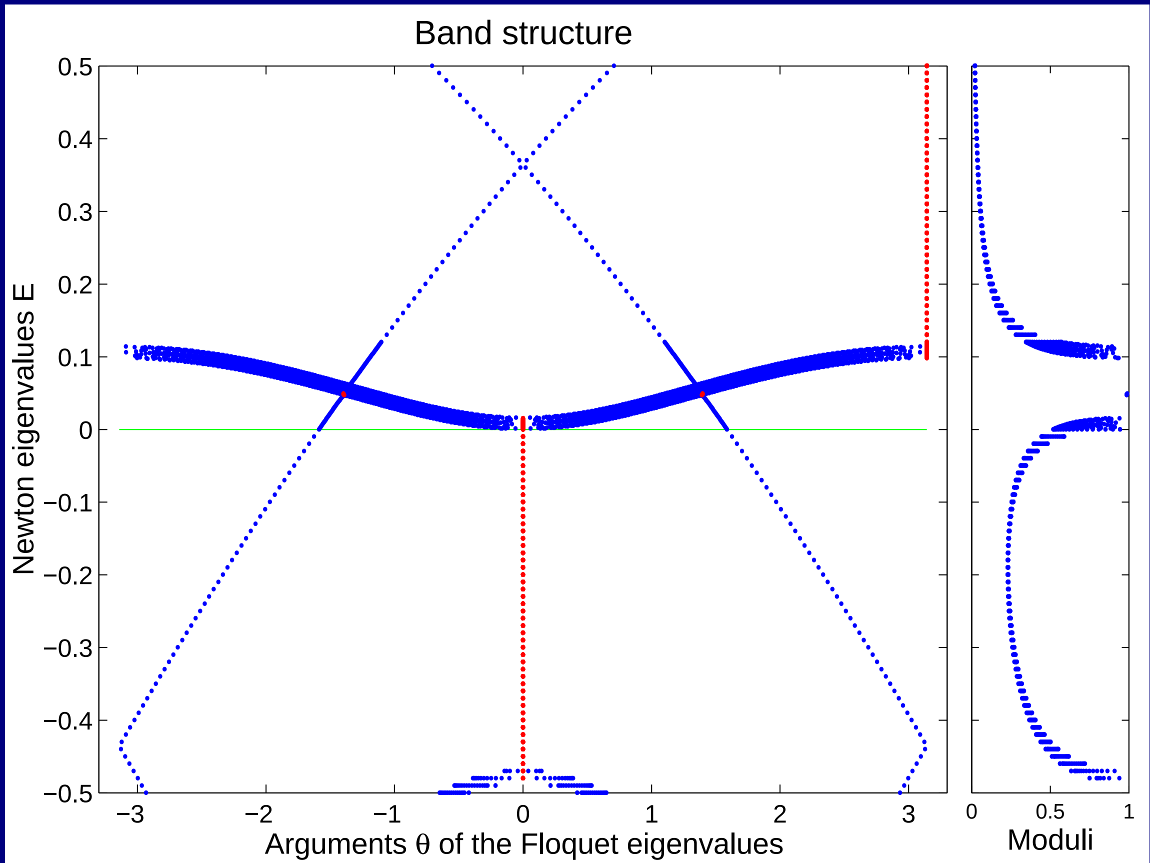
Example of bands at $\varepsilon = 0$

Cubic potential, zero coupling



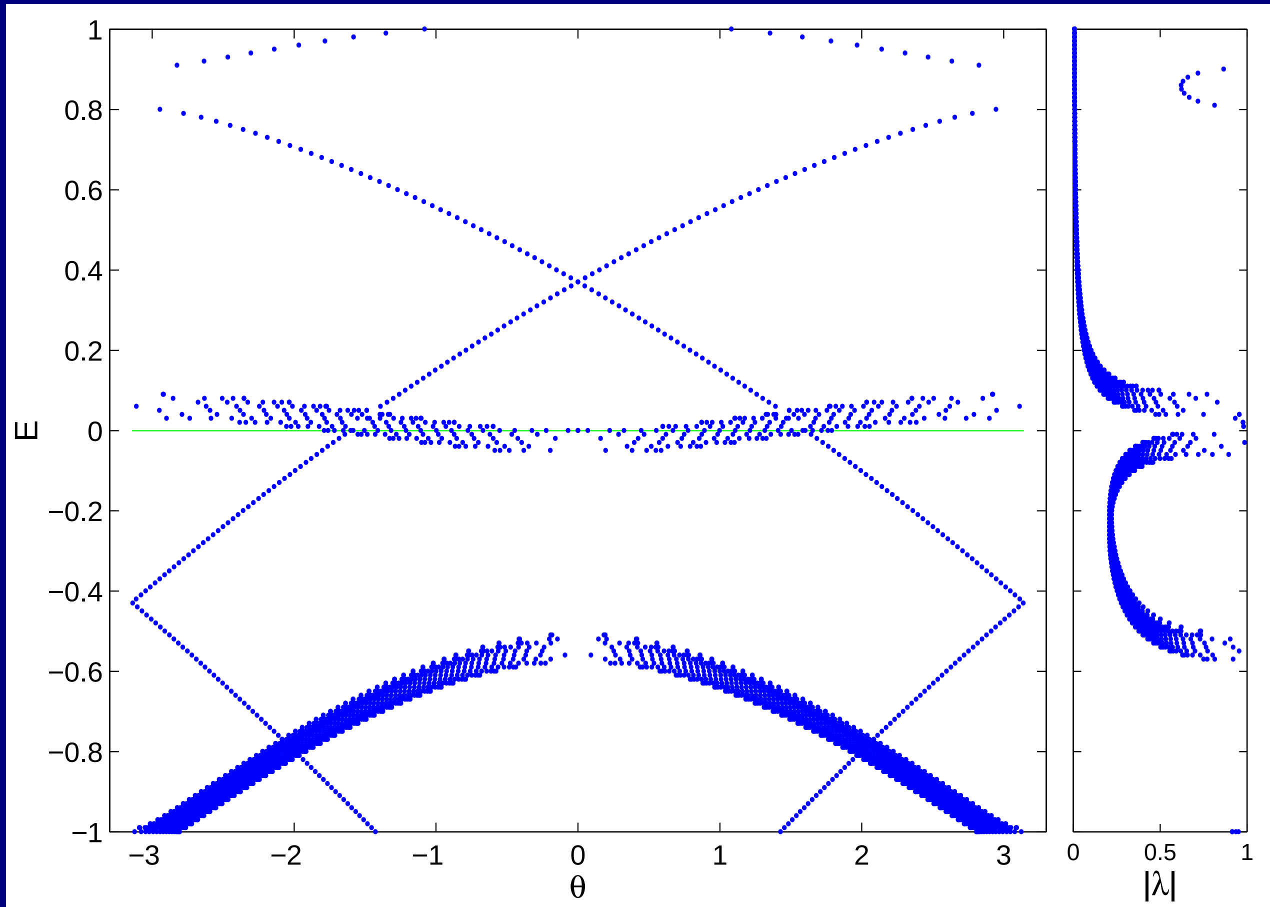
Example of unstable bands

Dark breather. Cubic potential, attractive coupling $\varepsilon = 0.004$



Example of stable bands

Dark breather. Cubic potential, repulsive coupling, $\varepsilon = 0.015$



Degenerate perturbation theory

If $\mathcal{N}_0$ is a linear operator with a degenerate eigenvalue E_0, with eigenvectors $\{|v_n\rangle\}$, which are ortonormal with respect to a scalar product, i.e., $\langle v_n|v_m\rangle = \delta_{n\,m}$, and if $\varepsilon\,\tilde{\mathcal{N}}$ is a perturbation of $\mathcal{N}_0$, with ε small; then, to first order in ε, the eigenvalues of $\mathcal{N}_0 + \varepsilon\,\tilde{\mathcal{N}}$ are $E_0 + \varepsilon\;\lambda_i$, with λ_i being the eigenvalues of the perturbation matrix Q with elements $Q_{n\,m} = \langle v_n|\tilde{\mathcal{N}}|v_m\rangle$.

Scalar product $\quad \langle \xi_1|\xi_2\rangle = \sum_{n=1}^{N}\int_{-T/2}^{T/2}\xi_1^*(t)\xi_2(t)\,\mathrm{d}t$

Basis $N-p$ $\quad$ elements (excited oscillators):

$$|n\rangle = \frac{1}{\mu}\begin{bmatrix}\vdots\\ 0\\ \dot{u}^0\\ 0\\ \vdots\end{bmatrix} \quad ; \quad \mu = \sqrt{\int_{-T/2}^{T/2}(\dot{u}^0)^2\,\mathrm{d}t}$$

Operators

$$\mathcal{N}_\varepsilon(u)\,|\xi\rangle = \overbrace{|\ddot{\xi}\rangle + V''(u)\cdot|\xi\rangle}^{\mathcal{N}_0\,|\xi\rangle} + \varepsilon\,\overbrace{\Big(V'''(u)\cdot u_\varepsilon\cdot|\xi\rangle + C\,|\xi\rangle\Big)}^{\tilde{\mathcal{N}}\,|\xi\rangle} =$$
$$= (\underbrace{E_0}_{0} + \varepsilon\,\lambda_i)|\xi\rangle$$

Demonstration (1)

Deriving with respect to ε, at $\varepsilon = 0$, the dynamical equations we obtain:

$$|\ddot{u}_\varepsilon\rangle + V''(u)\cdot|u_\varepsilon\rangle + C\,|u\rangle = 0 \quad \text{or} \quad N_0|u_\varepsilon\rangle = -C\,|u\rangle\,. \tag{1}$$

$\tilde{C}_{n\,m} = \langle n|C|m\rangle$ is C without the columns and rows corresponding to the oscillators at rest.

$$\begin{aligned} \langle n|V'''(u)\cdot u_\varepsilon|m\rangle &= \\ \frac{1}{\mu^2}\int_{-T/2}^{T/2}[\dots,0,\dot{u}_n^0,0,\dots]\quad[\dots,0,V'''(u^0)\,u_{m,\varepsilon}\,\dot{u}_m^0,0,\dots]^\dagger\,\mathrm{d}t &= \\ \frac{\delta_{n\,m}}{\mu^2}\int_{-T/2}^{T/2}\dot{u}^0\,V'''(u^0)\,u_{n,\varepsilon}\,\dot{u}^0\,\mathrm{d}t\,, \end{aligned} \tag{2}$$

with $u_{n,\varepsilon} = \left(\frac{\partial u_n}{\partial\varepsilon}\right)_{\varepsilon=0}$. Thus, $\langle n|V'''(u)\cdot u_\varepsilon\cdot|n\rangle = 0$ if $n \neq m$

To calculate the last integral in (2) we will integrate by parts and use that the integral in a period of the derivative of a periodic function is zero. Besides, the functions $u_{n,\varepsilon}$ are periodic as the coefficients of their Fourier series are given by the derivatives with respect to ε of the Fourier coefficients of u_n. In the deduction below, all the integral limits are $-T/2$ and $T/2$, and the terms between brackets from integration by parts will be zero.

Demonstration (2)

The last integral in Eq. (2) becomes:

$$
\begin{aligned}
\left[\dot u^0\, u_{n,\varepsilon}\, V''(u^0)\right]_{-T/2}^{T/2} - \int V''(u^0)\dot u^0\, \dot u_{n,\varepsilon}\,\mathrm{d}t - \int V''(u^0)\, \ddot u^0\, u_{n,\varepsilon}\,\mathrm{d}t \;&= \\
-\left[V'(u^0)\,\dot u_{n,\varepsilon},\right]_{-T/2}^{T/2} + \int V'(u^0)\,\ddot u_{\varepsilon,n},\mathrm{d}t - \int V''(u^0)\,\ddot u^0\, u_{n,\varepsilon}\,\mathrm{d}t \;&= \\
- \int \ddot u^0\,(\ddot u_{n,\varepsilon} + V''(u^0)\,u_{n,\varepsilon})\mathrm{d}t & \qquad (3)
\end{aligned}
$$

The term between parentheses, is the n component of the lhs of Eq. (1), i.e., it becomes $-\sum_m C_{n\,m}\,u^0_m$, where $u^0_m = u^0$, if the oscillator m is excited, and zero otherwise, i.e., it is $-\sum_m \tilde C_{n\,m}\,u^0 = -(\sum_m \tilde C_{n\,m})\,u^0$. Equation (3) becomes:

$$
\begin{aligned}
(\sum_m \tilde C_{n\,m}) \int \ddot u^0\, u^0\, dt \;&= \\
(\sum_m \tilde C_{n\,m})\left(\left[\dot u^0\, u^0\right]_{-T/2}^{T/2} - \int (\dot u^0)^2\, dt\right) \;&=\; -(\sum_m \tilde C_{n\,m})\mu^2\,. \qquad (4)
\end{aligned}
$$

That is, equation (2), leads to:

$$
\langle n|V'''(u)\cdot u_\varepsilon\cdot|n\rangle = -\sum_m \tilde C_{n\,m} \quad \text{at} \quad \varepsilon = 0\,. \qquad (5)
$$

Demonstration (3)

Therefore the diagonal elements of the perturbation matrix Q are

$$Q_{n\,n} = -\sum_{m} \tilde{C}_{n\,m} + \tilde{C}_{n\,n} = -\sum_{\forall\, m\neq n} \tilde{C}_{n\,m} = -\sum_{\forall\, m\neq n} Q_{n\,m}\,. \tag{6}$$

To summarize, the perturbation matrix Q is given by:

$$Q_{n\,m} = \tilde{C}_{n\,m}, \quad n \neq m \quad , \quad Q_{n\,n} = -\sum_{\forall\, m\neq n} Q_{n\,m}\,, \tag{7}$$

$\tilde{C}$ being the coupling matrix without the p rows and columns corresponding to oscillators at rest.

Perturbation matrix. Oscillators in phase

Modified coupling matrix $\tilde{C}$ Identical to C but without the rows and columns for the rest oscillators

Perturbation matrix Q

$$\begin{array}{rcl} Q_{n\,m} & = & \tilde{C}_{n\,m} \quad , \quad n \neq m \\ Q_{n\,n} & = & -\sum_{\forall\, m \neq n} Q_{n\,m} \end{array}$$

Example : 3–site breather. Code $[\,1, 1, 1]$. Elastic attractive interaction

$$C = \begin{bmatrix} 2 & -1 & 0 \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 \cdots & 0 & -1 & 2 \end{bmatrix} ; \tilde{C} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} ; Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$

$\lambda_1 = 0$: phase mode

λ_2, λ_3 positive

Conclusion If $\varepsilon > 0$ $\begin{array}{r} \text{Unstable for } V \text{ soft} \\ \text{Stable for } V \text{ hard} \end{array}$

Perturbation matrix. Oscillators not in phase. V symmetric

Code matrix $\sigma = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{bmatrix}$

Perturbation matrix Q

$$
\begin{aligned}
Q_{n\,m} &= \left\{ \begin{array}{ccc} \tilde{C}_{n\,m} & \text{if} & \sigma_n = \sigma_m \\ -\tilde{C}_{n\,m} & \text{if} & \sigma_n \neq \sigma_m \end{array} \right\} \quad n \neq m \\
Q_{n\,n} &= -\sum_{\forall\, m \neq n} Q_{n\,m}
\end{aligned}
$$

Symmetric multibreathers stability theorem

Let be $V(u_n)$ symmetric and $\{\lambda_i\}$ the eigenvalues of Q_{nm}:
(a) if $V(u_n)$ is hard and there is any negative value in $\{\epsilon\lambda_i\}$ the multibreather at low coupling will be unstable, and stable otherwise.
(a) if $V(u_n)$ is soft and there is any positive value in $\{\epsilon\lambda_i\}$ the multibreather at low coupling will be unstable, and stable otherwise.

Example : 3–site breather. Code $[-1, 1, -1]$. Elastic attractive coupling.

$$C = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} ; \; \tilde{C} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} ; \; Q = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0$ (phase mode), $\lambda_2 = -1$, $\lambda_3 = -3$ (both negative)
Conclusion: If $\varepsilon > 0$: Stable for V soft, Unstable for V hard.

Non–symmetric on–site potentials

Symmetry coefficient γ : $\gamma = -\dfrac{\displaystyle\int_{-T/2}^{T/2} \dot{u}^0(t)\, \dot{u}^0(t+T/2)\, \mathrm{d}t}{\displaystyle\int_{-T/2}^{T/2} \dot{u}^0(t)\, \dot{u}^0(t)\, \mathrm{d}t}$

Properties of γ

1) $\gamma = \gamma(\omega_\mathrm{b})$
2) $0 < \gamma < 1$
3) If V is symmetric $\gamma = 1$
4) $\omega_\mathrm{b} \longrightarrow \omega_0 \quad \Rightarrow \quad \gamma \longrightarrow 1$
5) Numerically: Fourier coefficients
6) Analytically: Morse potential: $\gamma = \omega_\mathrm{b}$

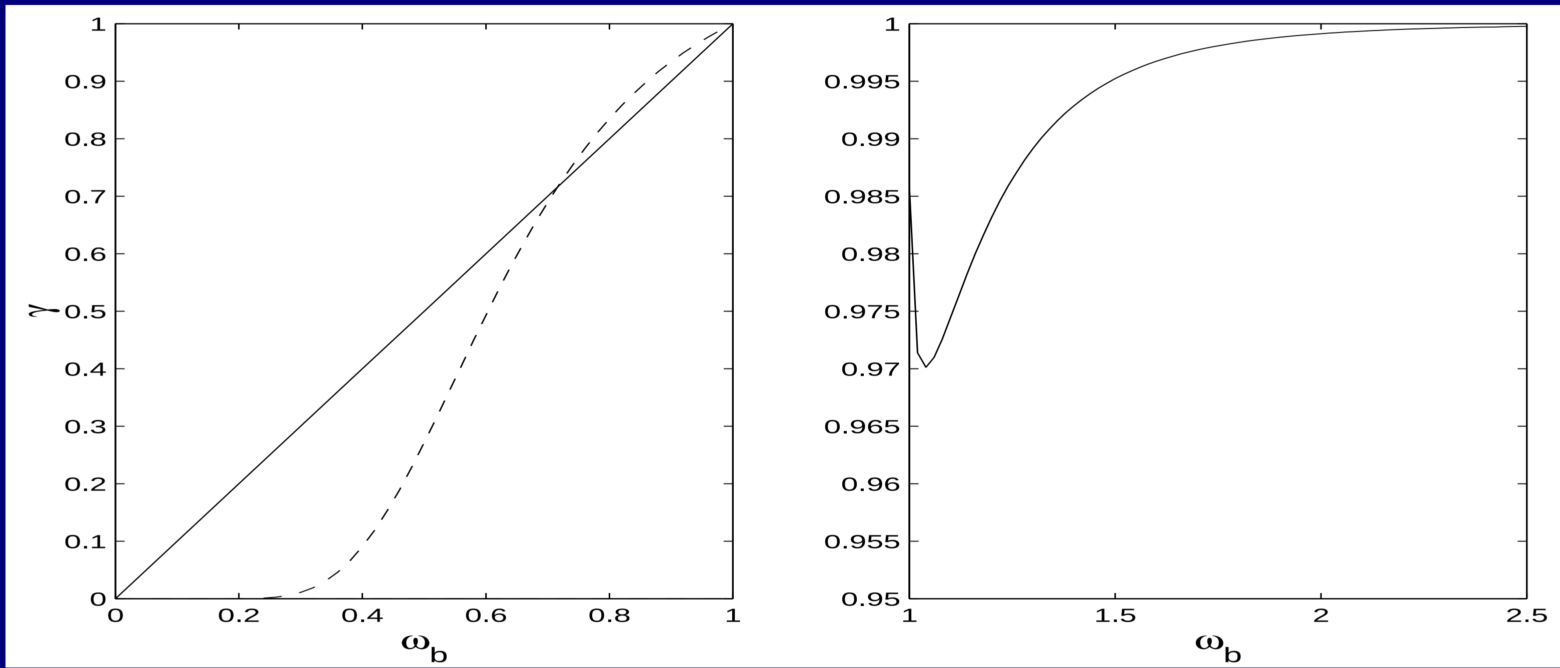
Perturbation matrix Q

$$Q_{n\,m} = \left\{ \begin{array}{ccc} \tilde{C}_{n\,m} & \text{if} & \sigma_n = \sigma_m \\ & & \\ -\gamma\, \tilde{C}_{n\,m} & \text{if} & \sigma_n \neq \sigma_m \end{array} \right\} \quad n \neq m$$

$$Q_{n\,n} = -\sum_{\forall\, m \neq n} Q_{n\,m}$$

Non–symmetric multibreather stability theorem

The symmetry coefficient γ versus $\omega_{\rm b}$



Soft

– – – – $V(u_n) = \frac{1}{2}u_n^2 - \frac{1}{3}u_n^3$

——— $V(u_n) = \frac{1}{2}(\exp(-u_n) - 1)^2$

Hard

$$V(u_n) = \frac{1}{2}u_n^2 + \frac{1}{3}u_n^3 + \frac{1}{4}u_n^4$$

Rest frequency $\omega_0 = 1$

Application. 2-site breathers

$$C = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Code $\pm[1,1]$: $\tilde{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$; $Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; $(\lambda_1, \lambda_2) = (0, +2)$

Code $\pm[-1,1]$: $\tilde{C} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $Q = \begin{bmatrix} -\gamma & \gamma \\ \gamma & -\gamma \end{bmatrix}$; $(\lambda_1, \lambda_2) = (0, -2\gamma)$

$\pm[1,\ 1]$	Attractive	Repulsive
Soft	Unstable	Stable
Hard	Stable	Unstable

$\pm[1, -1]$	Attractive	Repulsive
Soft	Stable	Unstable
Hard	Unstable	Stable

Theorem by Aubry for the symmetric case in:
JL Marín. *PhD thesis*, June 1997.

Application. 3–site breathers

Code $\pm [1,1,1]$:

	Attractive coupling	Repulsive coupling
Soft	Unstable	Stable
Hard	Stable	Unstable

Code $\pm [-1,1,-1]$:

	Attractive coupling	Repulsive coupling
Soft	Stable	Unstable
Hard	Unstable	Stable

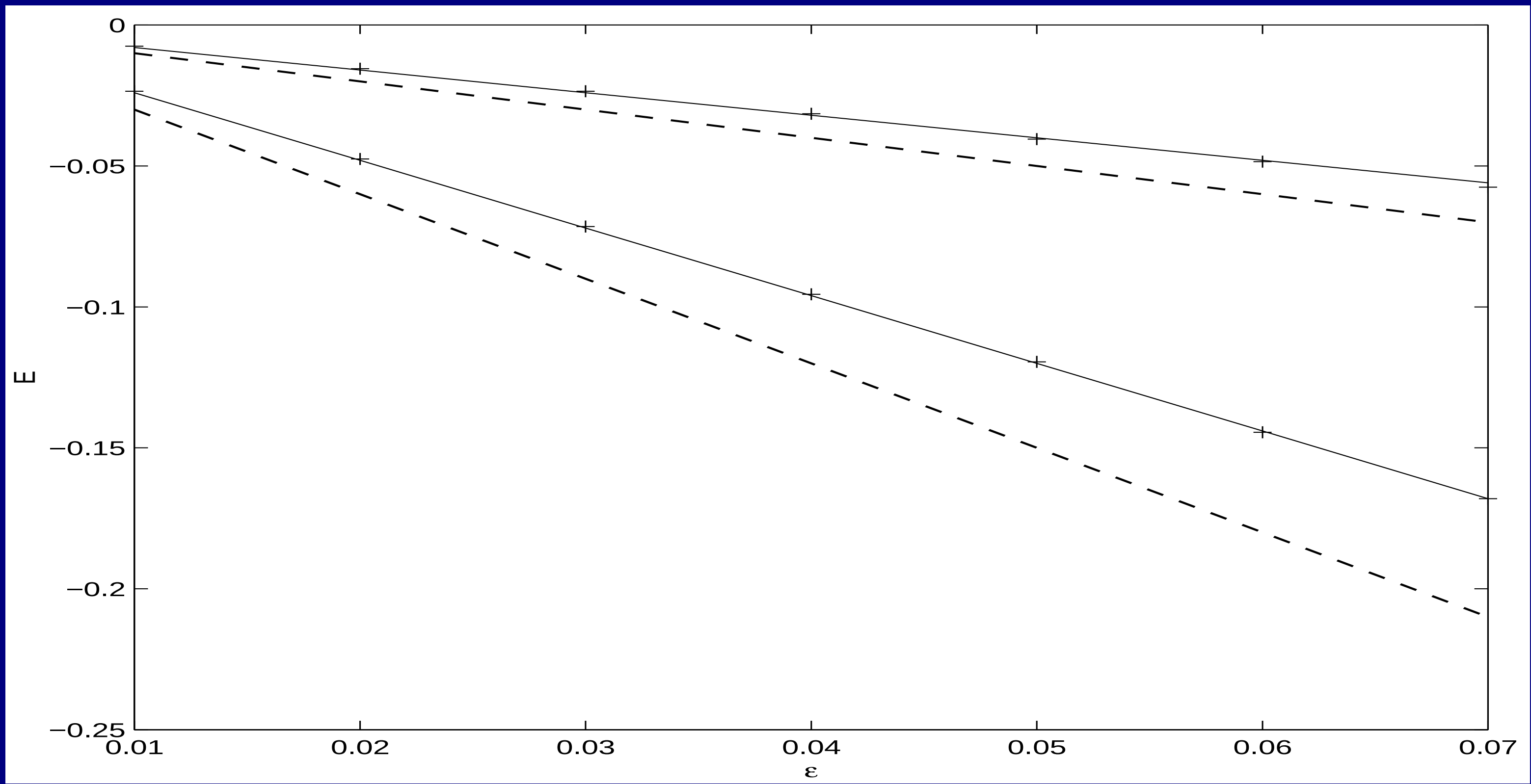
Code $\pm [1,1,-1]$:

Always unstable

MST versus *exact* numerical

Morse potential. Elastic attractive coupling. Code $[-1, 1, -1]$.
Frequency $\omega_{\mathrm{b}} = \gamma = 0.8$.

Eigenvalues E of the Newton operator versus the coupling ε



$+$ Numerical. $----$ Symmetric MST. ——— Non–symmetric MST

Multibreathers

Group of contiguous oscillators excited at $\varepsilon = 0$

In–phase (wave vector $q = 0$) :

	Attractive coupling	Repulsive coupling
Soft	Unstable	Stable
Hard	Stable	Unstable

Out–of–phase (wave vector $q = \pi$) :

	Attractive coupling	Repulsive coupling
Soft	Stable	Unstable
Hard	Unstable	Stable

Mixed : Always unstable

Phonobreathers

All oscillators excited at zero coupling

- With **free–ends** or **fixed–ends** boundary conditions: as multibreathers

  Coherent with:

  I. Daumont, T. Dauxois, and M. Peyrard.
  Nonlinearity, 10:617–630, 1997. (Modulational instability).

  AM Morgante, M Johansson, G Kopidakis, and S Aubry.
  Phys. D, 162:53, 2002. (DNLS)

  S Aubry. *Physica D*, 103:201–250, 1997. (Theorem 9, action properties)

- With **periodic** boundary conditions: **Parity instabilities**

Dark breathers

- **Free–ends** of **fixed–ends**: almost as multibreathers:
  There is a degenerate 0-eigenvalue of Q $\quad\Rightarrow$
  If the eigenvalues of Q correspond to instability: instability
  If the eigenvalues of Q correspond to stability: undefined
- Periodic boundary conditions: **Parity instabilities**
- **Parity instabilities. Example.**
  Dark breather. Attractive coupling. Soft on-site potential.
  Code $\sigma = [\,\cdots, 1, -1, 0, 1, -1, \cdots]$

  N **odd** ($N = 5$):
  $\sigma = [\,1, -1, 0, 1, -1]$: **stable**

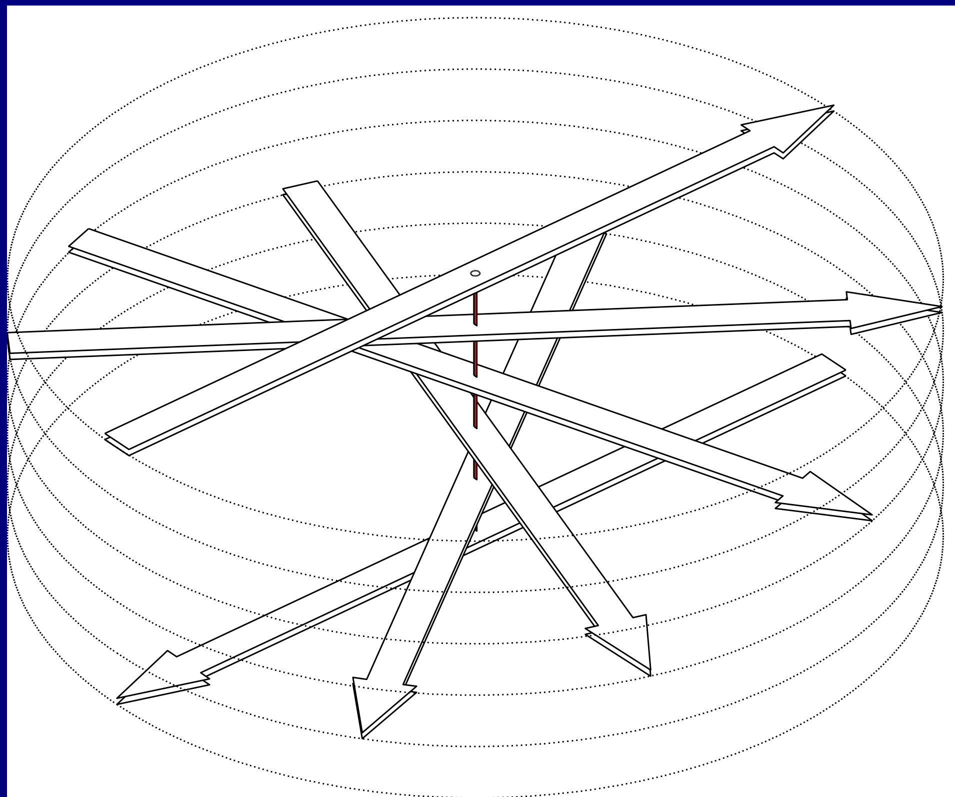
  N **even** ($N = 6$): $\sigma = [\,1, -1, 0, 1, -1, 1]$
  equivalent to $\sigma = [-1, 0, 1, -1, 1, 1]$: **unstable**.

  But with code $\sigma = [\,\cdots, -1, 1, 0, 1, -1, \cdots]$ the conditions are reversed.

A model with LRI (1)

Twist model for DNA:

B Sánchez-Rey, JFR Archilla, F Palmero, and FR Romero.
Phys. Rev. E, 66:017601–017604, 2002.
Selected by Virtual Journal of Biological Physics Research 4(1), 2002.



A model with LRI (2)

Dynamical equations

$$\ddot{u}_n + V'(u_n) + \varepsilon \sum_{m=n-N/2}^{n+N/2} \frac{\cos[\theta_{tw}(n-m)]}{|n-m|^3} u_m = 0$$

$V(u_n)$: Morse potential

Coupling matrix $C_{n\,m} = \dfrac{\cos[\theta_{tw}(n-m)]}{|n-m|^3}$

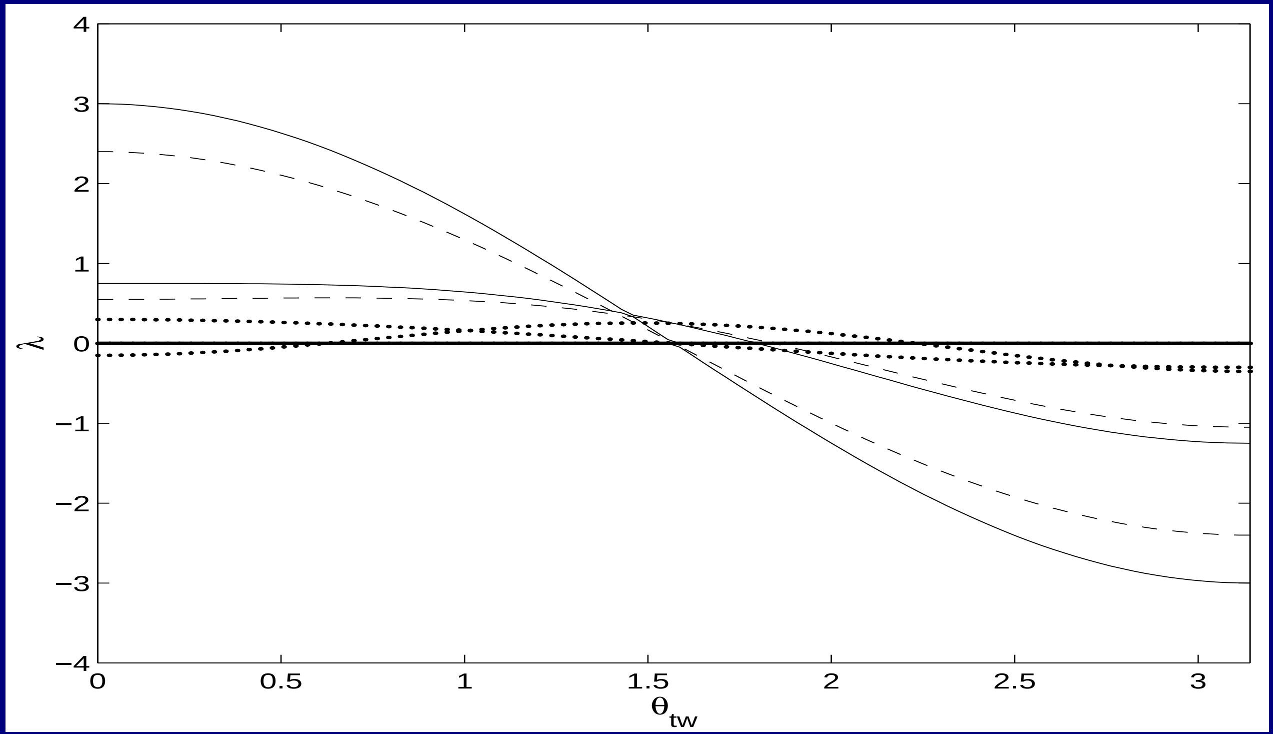
Some results

Code	$\theta_{tw} = 0$	$\theta_{tw} = \pi$
11	Stable	Unstable
1 -1	Unstable	Stable
101	Stable	Stable
111	Stable	Unstable
1-1 1	Unstable	Stable
11-1	Unstable	Stable

A model with LRI (3)

Dependence on the symmetry coefficient $\gamma = \omega_{\rm b}$. Code $[-1, 1, -1]$.

Eigenvalues λ_i of Q versus the twist angle θ_{tw}



———: $\gamma = \omega_{\rm b} = 1$ $\quad$ - - - -: $\gamma = \omega_{\rm b} = 0.8$ $\quad$ · · · ·: $\gamma = \omega_{\rm b} = 0.1$

Instability induced by the nonlinearity for $\gamma = 0.1$ and $\theta_{tw} = 2\,$rad.

Multibreathers in the Peyrard–Bishop DNA model

Dynamical equations Planar model

$$\ddot{u}_n + V'(u_n) + \epsilon\,(2\,u_n - u_{n-1} - u_{n+1}) = 0$$

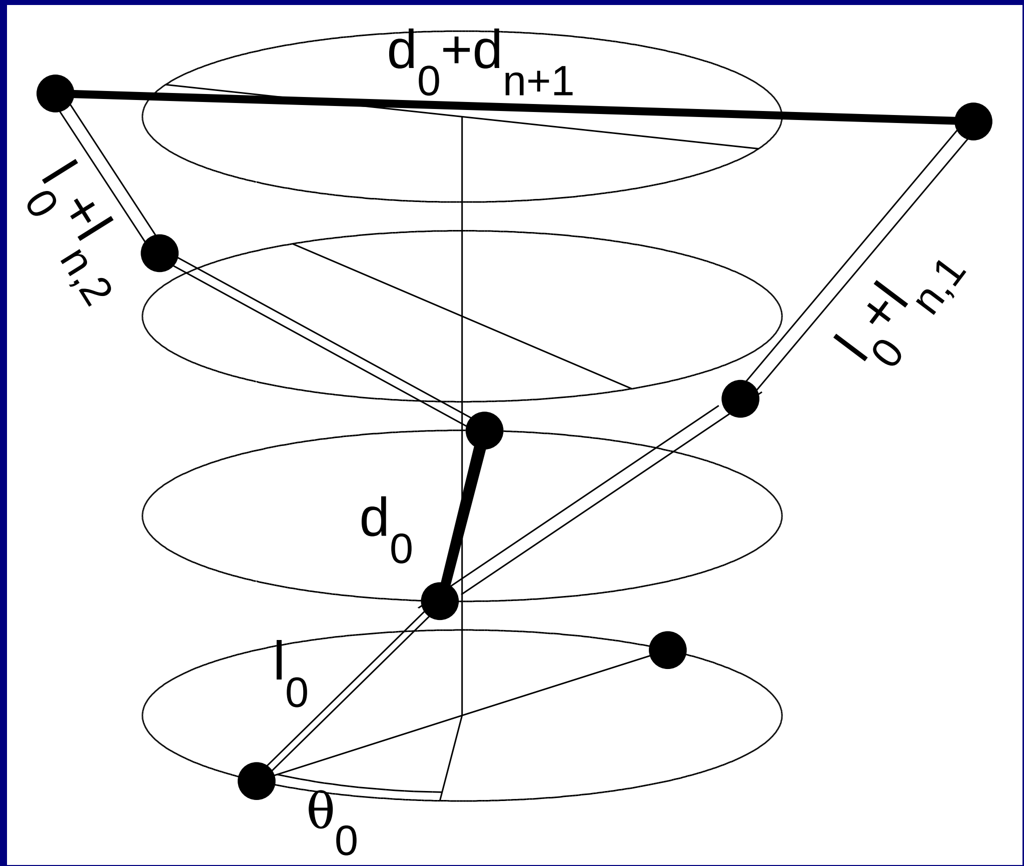
V: Morse potential (soft) and attractive coupling

In-phase multibreathers are unstable

Helical model for DNA

- D Hennig and JFR Archilla, *Multi-site H-bridge breathers in a DNA–shaped double strand.* Physica Scripta, 69(2):150-160, 2004.
- Variant of:
  M. Barbi, S. Cocco and M. Peyrard, *Phys. Lett. A* **253**, 358 (1999). And continuations.

A helical DNA–like model



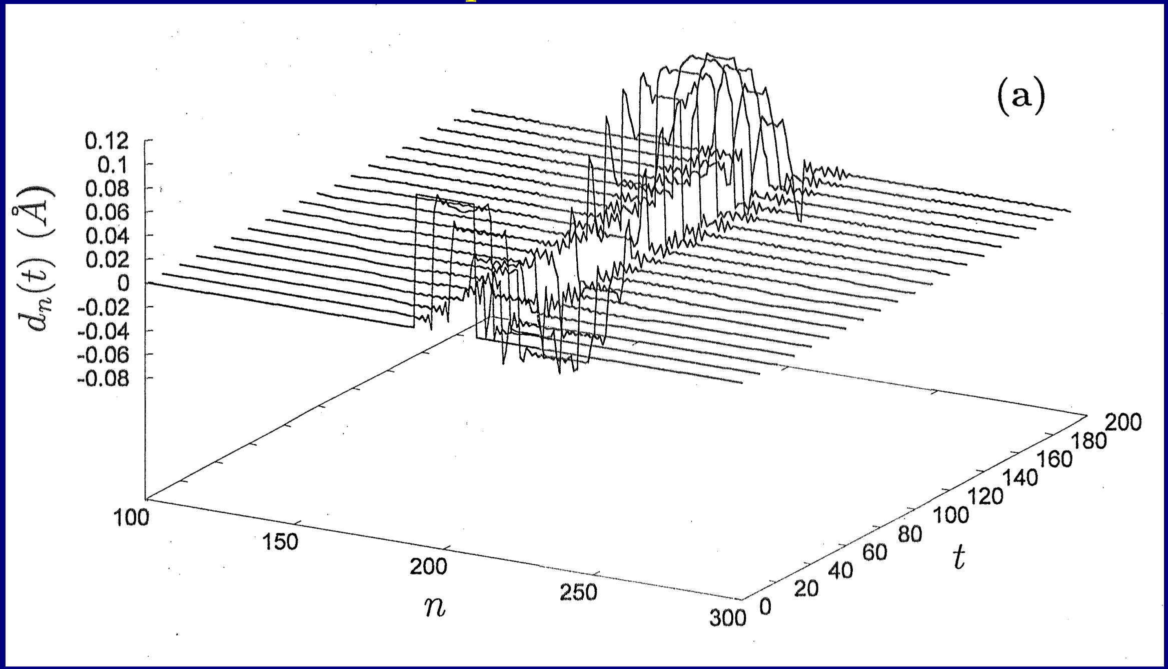
Multibreathers in the helical DNA–like model

Effective repulsive coupling

Linear equations

$$\ddot{r}_n = -\omega_0^2\, r_n - \varepsilon\, \sin^2(\frac{\theta_0}{2})\,(2\, r_n + r_{n+1} + r_{n-1}) - \varepsilon\, \sin(\frac{\theta_0}{2})\cos(\frac{\theta_0}{2})\, d_0\,(\alpha_{n+1} - \alpha_{n-1})\,,$$

Stable in–phase multibreathers



The helical shape supports in–phase, stable multibreathers

Generalization

Hamiltonian $\quad H = \sum_n \left(\frac{1}{2} m_n \dot{u}_n^2 + V_n(u_n)\right) + \varepsilon W(u)$

Dynamical equations $\quad m_n \ddot{u}_n + V_n'(u_n) + \varepsilon \dfrac{\partial W(u)}{\partial u_n} = 0$

$V_n(u_n)$ heterogeneous
multiple wells, etc

Given periodic solution at $\varepsilon = 0$: $\quad u^0 = \begin{bmatrix} u_1^0 \\ \vdots \\ u_N^0 \end{bmatrix}$

u_n^0 determined by
n, well, phase
(non time–reversible)

Basis elements : $\quad |n\rangle = \dfrac{1}{\mu_n} \begin{bmatrix} \vdots \\ 0 \\ \dot{u}_n^0 \\ 0 \\ \vdots \end{bmatrix} \quad ; \quad \mu_n = \sqrt{\int_{-T/2}^{T/2} (\dot{u}_n^0)^2 \mathrm{d}t}$

Perturbation matrix

$$Q_{n\,m} = \frac{1}{\mu_n\,\mu_m} \int_{-T/2}^{T/2} \dot{u}_n^0 \, \frac{\partial^2 W(u^0)}{\partial u_n \, \partial u_m} \, \dot{u}_m^0 \, \mathrm{d}\,t \, , \quad n \neq m \, ; \quad Q_{n\,n} = -\sum_{\forall\, m \neq n} \frac{\mu_m}{\mu_n} Q_{n\,m}$$

Generalized multibreathers stability theorem

Conclusions

1 A theory for calculating the stability of **any** multibreather in **any** Klein–Gordon system at (relatively) low coupling

2 For (relatively) simple systems is very simple

3 For complex systems is (relatively) complex

4 Application to a number of systems

2–site breathers	3–site breathers
Multibreathers	Phonobreathers
Dark breathers	Parity instabilities
Systems with LRI	Nonlinear instabilities

5 Potential consequences for helical DNA models

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J Cuevas, JFR Archilla and FR Romero.
Nonlinearity 18 (2005) 769-790.

Multibreather and vortex breather stability in Klein–Gordon lattices: Equivalence between two different approaches.
J Cuevas, V Koukouloyannis, PG Kevrekidis and JFR Archilla
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