Solitons in Schrödinger Lattices with Local Inhomogeneities

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Introduction

- Past few years interest in nonlinear discrete systems. Applications: optics of waveguide arrays, Bose-Einstein condensates, micromechanical models, biological systems (DNA, proteins…).
- One of the most prototypical models: The Discrete Nonlinear Schrödinger Equation (DNLS).
- DNLS with localized impurities: Nonlinearity and periodicity. Scattering phenomena and excitation of impurity modes. Applications: Superconductors, dynamics electron-phonon interactions. Propagation of light in super-lattices with defects and photonic crystals.
- Interesting studies in connection to the interplay of the localized modes with impurities.

The model

- One dimensional discrete system with a defect described by the DNLS equation:

$$i\frac{d\psi_n}{dt} + \gamma|\psi_n|^2\psi_n + C\Delta\psi_n + \alpha_n\psi_n = 0$$

- Transformation $\Psi_n \rightarrow \Psi_n e^{-2iCt}$

$$i\frac{d\psi_n}{dt} + \gamma|\psi_n|^2\psi_n + C(\psi_{n+1} + \psi_{n-1}) + \alpha_n\psi_n = 0$$

- Single point defect: $\alpha_n = \alpha\, \delta_{n,0}$. Renormalization parameters (γ=1, focusing case). Under staggering transformation, can be transformed to defocusing case with opposite sign of the impurity.
- Dynamical invariants: Hamiltonian and the norm $P = \sum_n |\psi_n|^2$

Stationary solutions

- Stationary solutions:

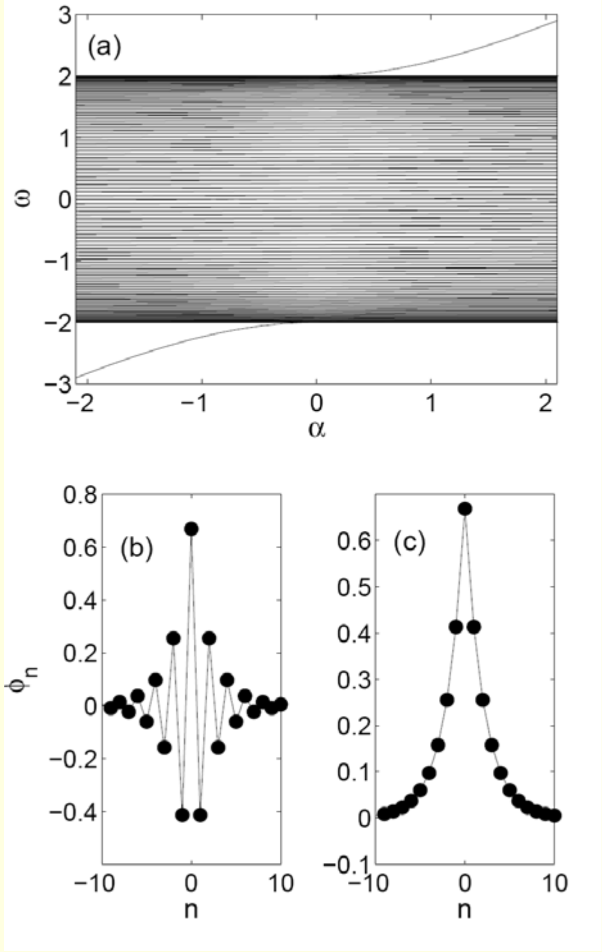
$$\psi_n = e^{i\omega t}\phi_n$$

- Equation:

$$-\omega\phi_n + C(\phi_{n+1} + \phi_{n-1}) + \phi_n^3 + \alpha_n\phi_n = 0$$

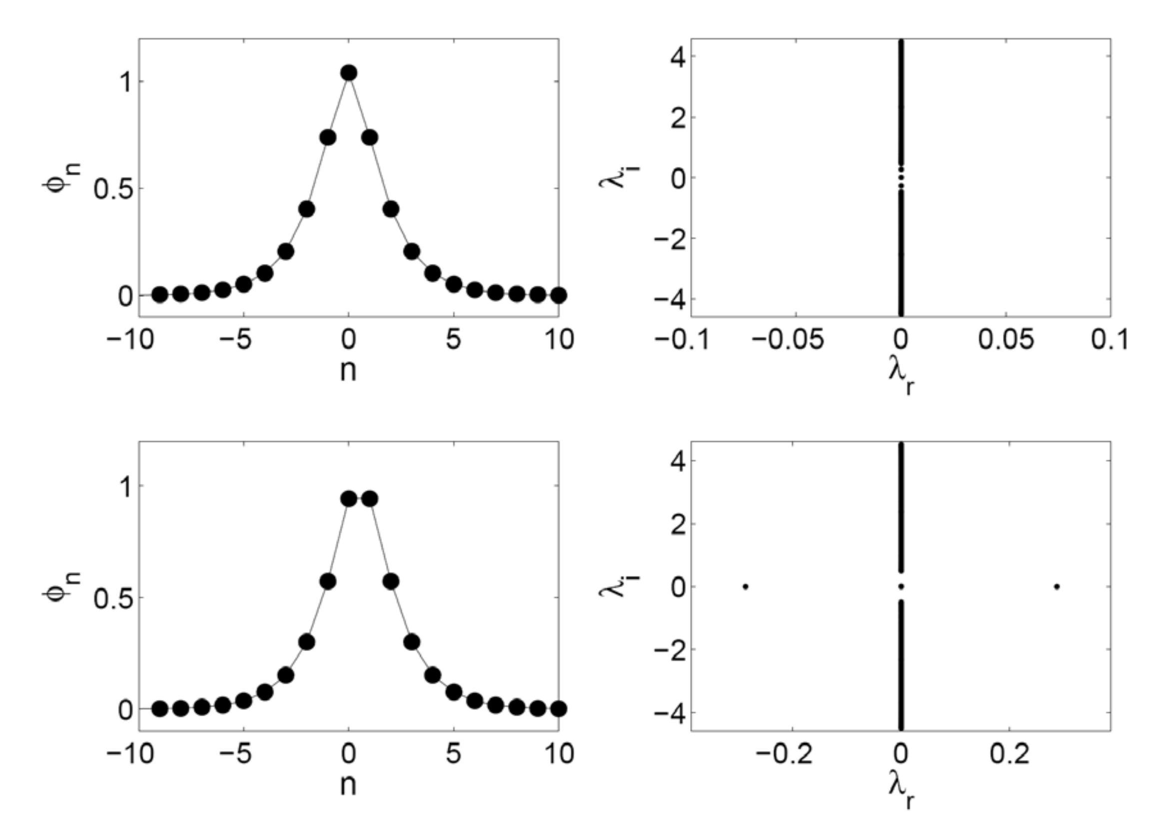
Linear modes

- Resolution of a eigenvalue problem.
- Number of sites large and periodic boundary conditions problem can be solved. *N*-1 extended modes with frequencies distributed in the interval [-2C,2C] and an impurity localized mode.



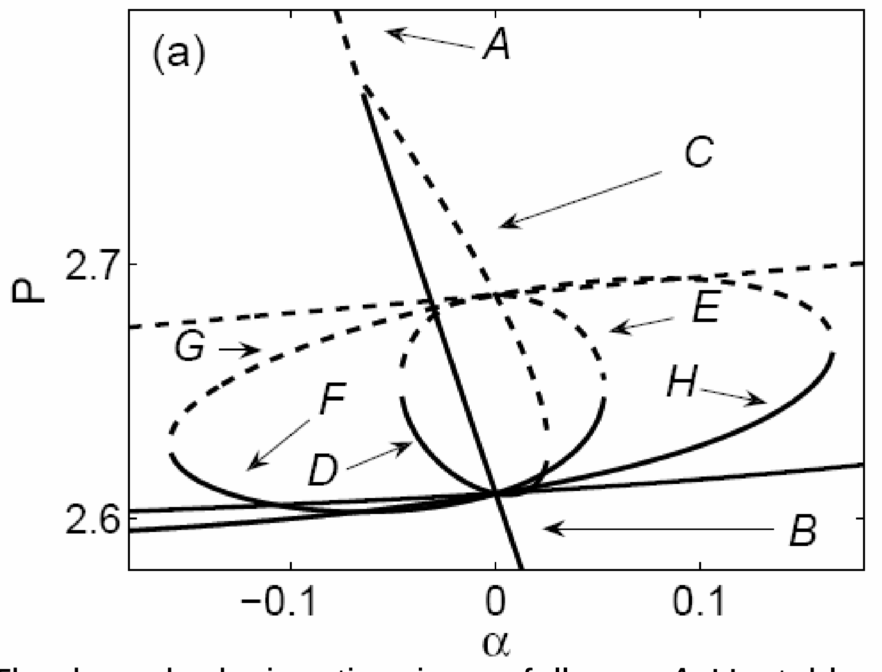
Nonlinear stationary states

- Anticontinuum limit (AC) and a Newton-Raphson fixed point algorithm.
- Standard stability analysis.
- Homogeneous system: Fundamental stationary modes exist centered either on a lattice site (stable) or between two adjacent lattice sites (unstable).



Bifurcations of stationary states

- In general, the stable on-site soliton localized at the impurity merges with the unstable inter-site centered one localized between impurity and its neighboring site (beyond some critical value of $|\alpha|$) and the resulting state becomes unstable.



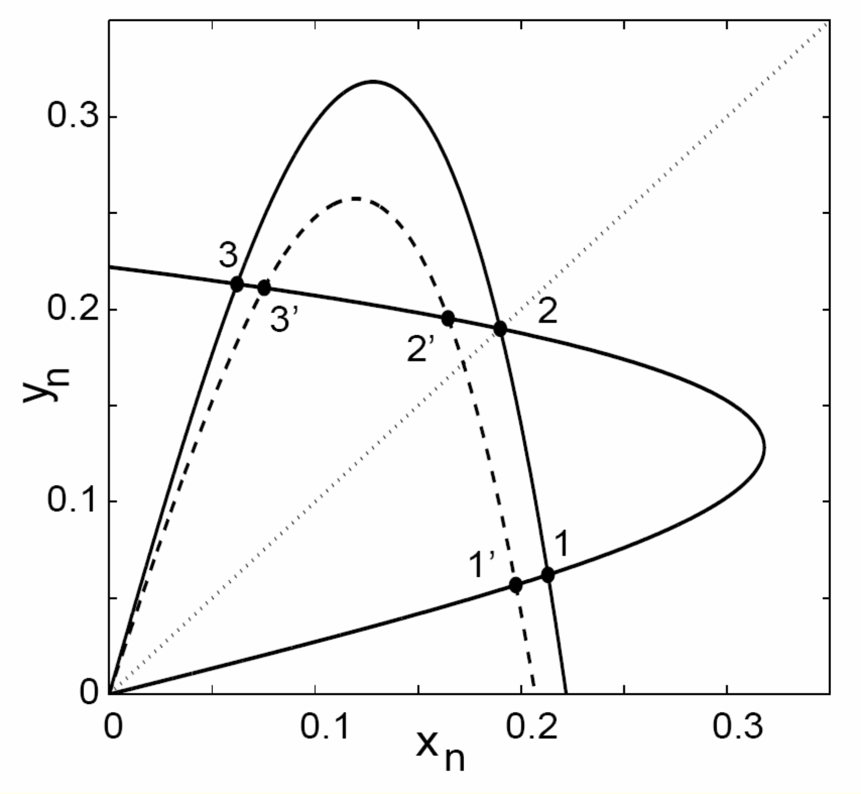
The branch designation is as follows: *A* Unstable soliton centered at the impurity ($n = n_0$), *B* stable on-site soliton centered at $n = n_0$, *C* Unstable inter-site soliton centered at $n = n_0 + 0{:}5$, *D* stable on-site soliton at $n = n_0 + 1$, *E* unstable inter-site soliton at $n = n_0 + 1{:}5$, *F* stable on-site soliton at $n = n_0 +2$, *G* unstable inter-site soli ton at $n = n_0+2{:}5$, and *H* stable on-site soliton at $n = n_0+3$.

Invariant manifold approximation

- Numerical and analytical approximate method (G. James, B. Sánchez-Rey and J. Cuevas. Preprint, 2007. arXiv:nlin.PS/0710.4114).
- Homogeneous system, the equation corresponding to stationary states can be recast as a two-dimensional map. Defining $y_n = \Phi_n$ and $x_n = \Phi_{n-1}$,

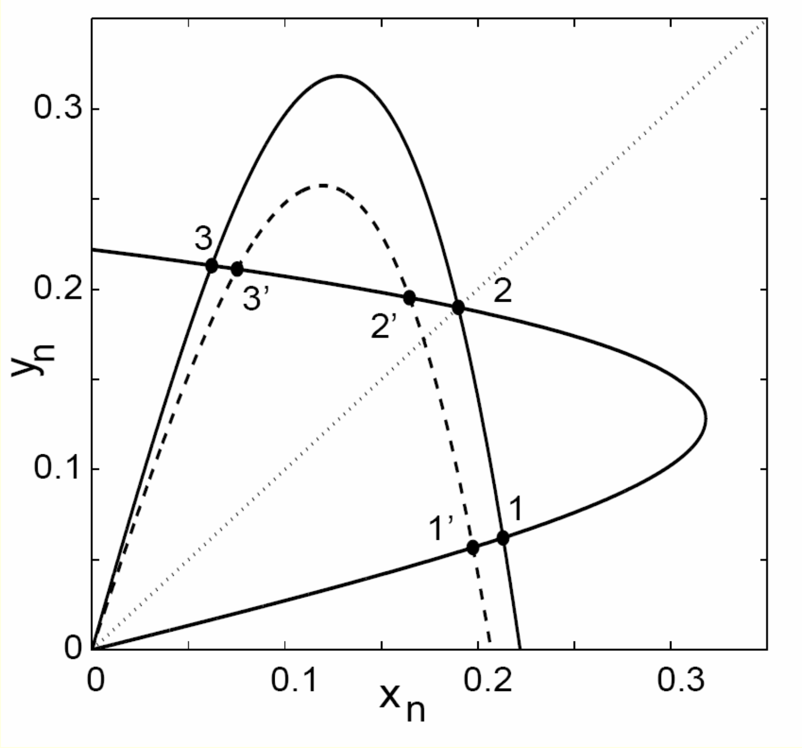
$$\begin{cases} x_{n+1} = y_n \\ y_{n+1} = (\omega y_n - y_n^3)/C - x_n \end{cases}$$

- The origin is a hyperbolic saddle point. There exist a one-a one-dimensional stable and a one-dimensional unstable emanating from the origin in two directions. These manifolds intersect in general transversally (infinity homoclinic orbits). On-site and intersite solitons correspond to the primary intersection points. Each intersection point defines a condition that allows to determine the soliton.

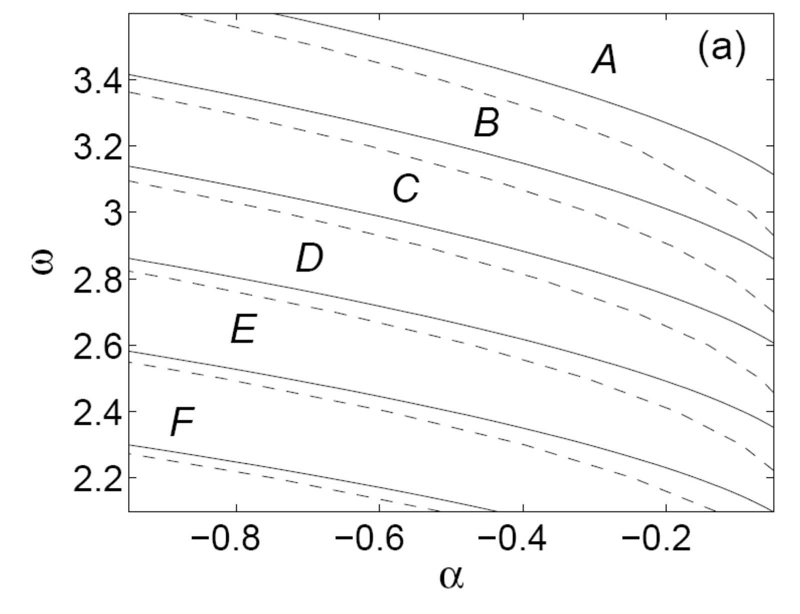


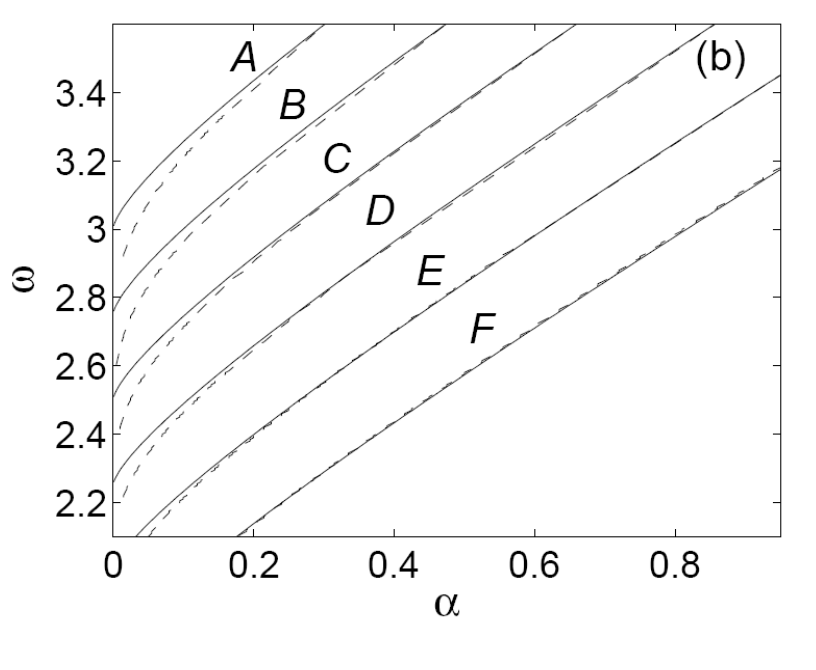
Effect of the inhomogeneity

- Effect of the inhomogeneity introduced as a linear transformation of the unstable manifold.
- When $\alpha>0$, the unstable manifold moves downwards, changing the intersections between the transformed unstable manifold and the stable manifold to points 1', 2' and 3'. For $\alpha=\alpha_c$, both manifolds become tangent. Thus, for $\alpha > \alpha_c$ intersections 3' and 2' are lost, that is, for $\alpha = \alpha_c$ the breathers centered at n=1 and n=0.5 experience a tangent bifurcation.
- On the contrary, if $\alpha<0$, intersections 1' and 2' are lost when $|\alpha|>|\alpha_c|$, leading to a bifurcation between the breathers centered at n=0.5 and n=0.
- Determination of α_c : Cubic approximation of the unstable manifold and tangent points.



Bifurcation loci (different coupling constant C)

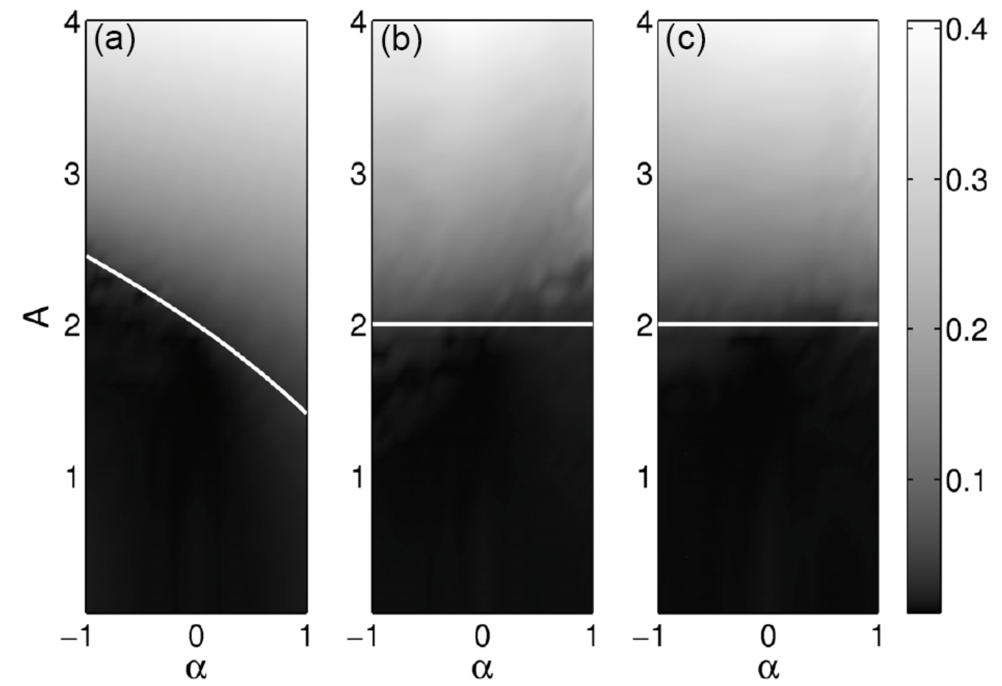




Threshold for solitary wave formation

- Solitary wave formation problem.
- Minimal amplitude threshold for a compactum of initial data to nucleate a localized mode
- Good approximation? (P.G. Kevrekidis et al. Phys. Lett. A, 372, 2247, 2008):

$$-\frac{A^4}{2} + (2C - \alpha_k)A^2 < 0$$



$$L(A) = \frac{\sum_n |\varphi_n|^2}{(\sum_n |\varphi_n|)^2}$$

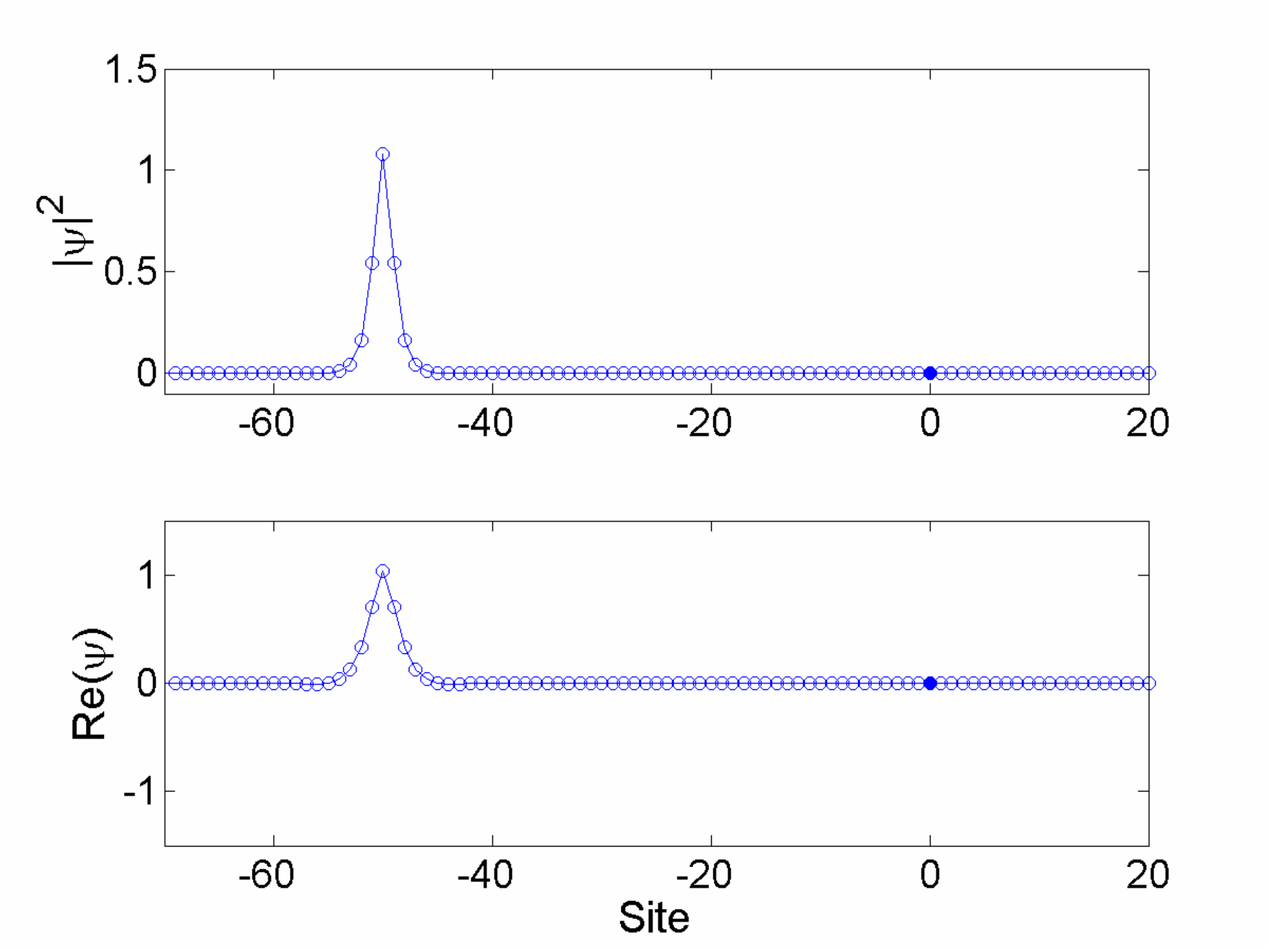
Interaction of a moving localized mode with an impurity

- Propagating localized modes, with weak radiative loses, and its interaction with the impurity.
- Generation of moving soliton: Perturbation of a stationary soliton by adding a thrust:

$$\varphi_n(t=0) = \phi_n e^{iqn}$$

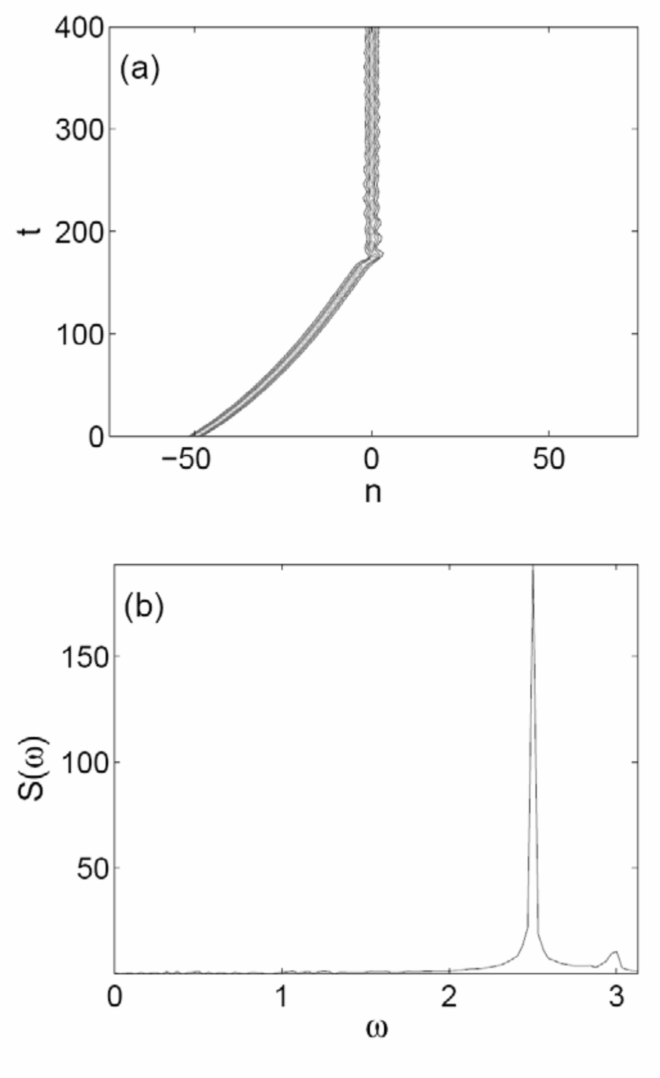
- In general, if *q* is large enough, the soliton moves with a small loss of radiation. We can essentially distinguish four fundamental regimes.

I. Trapping

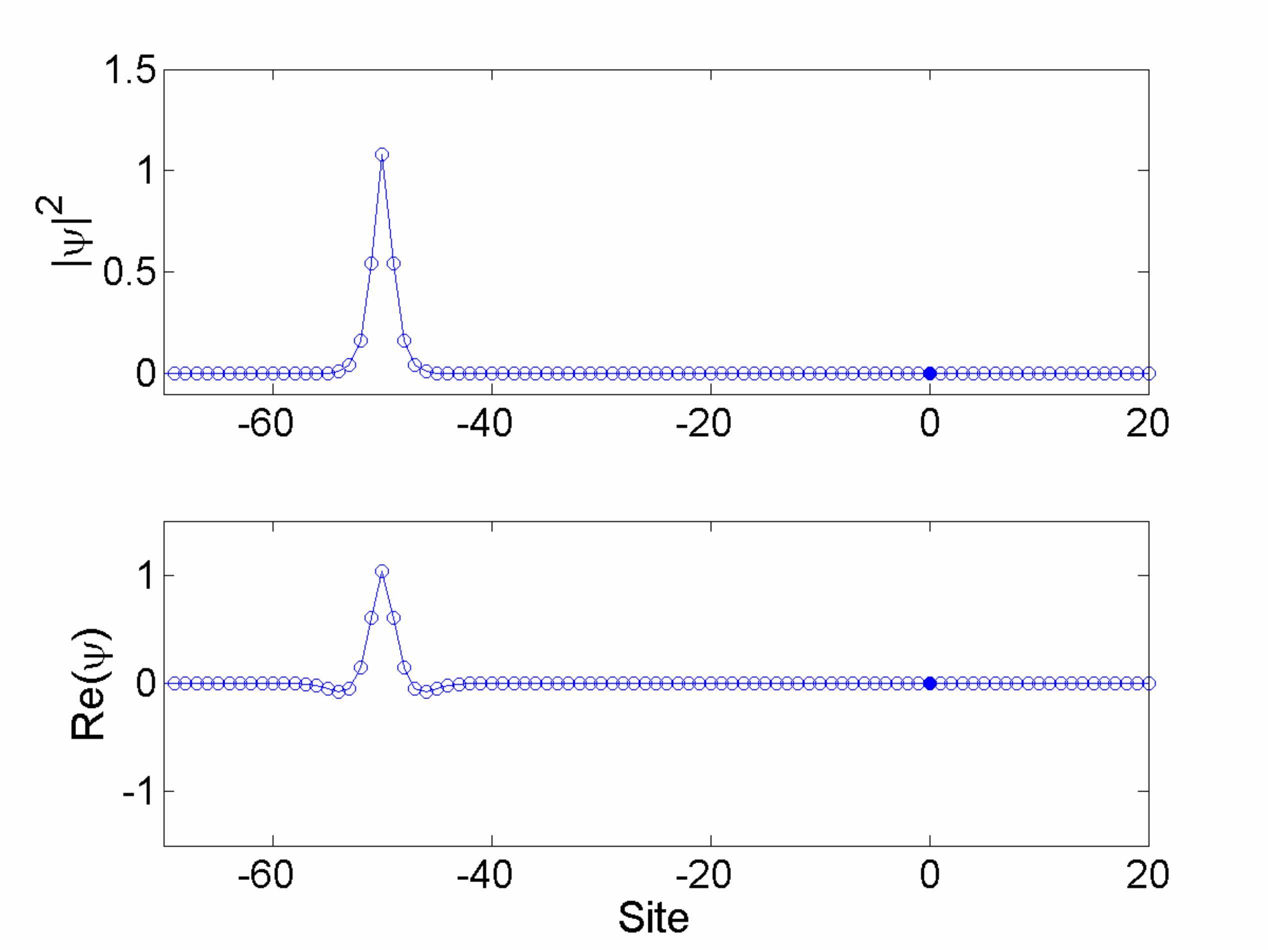


Trapping (II)

- Parameters α and q small enough. Atractive impurity.
- Only a small fraction of energy is lost by means of phonon radiation.
- In general, the frequency of the trapped soliton is slighty smaller than that of the incident soliton. Smaller energy and power than the corresponding nonlinear mode with the frequency of incident soliton.
- Conclusion: The incident breather can activate the nonlinear mode. Nearly all energy remains trapped.

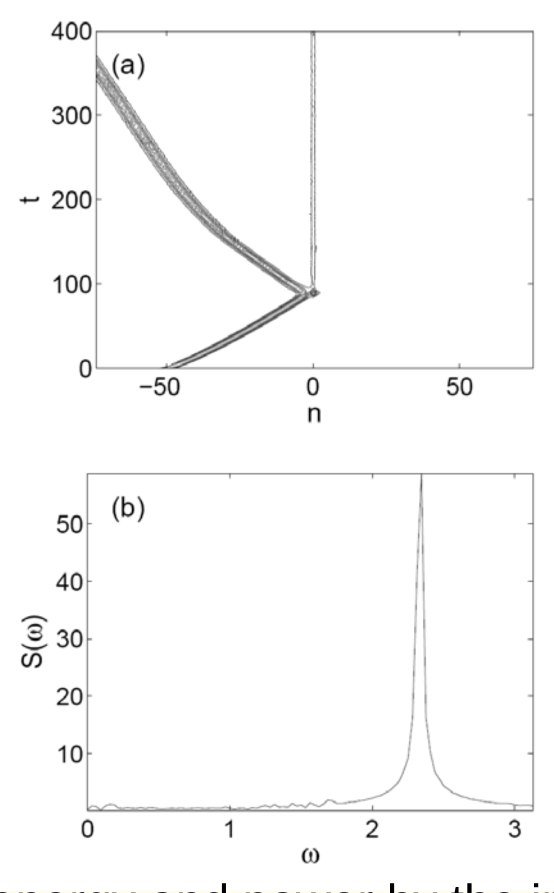


II. Trapping and reflection



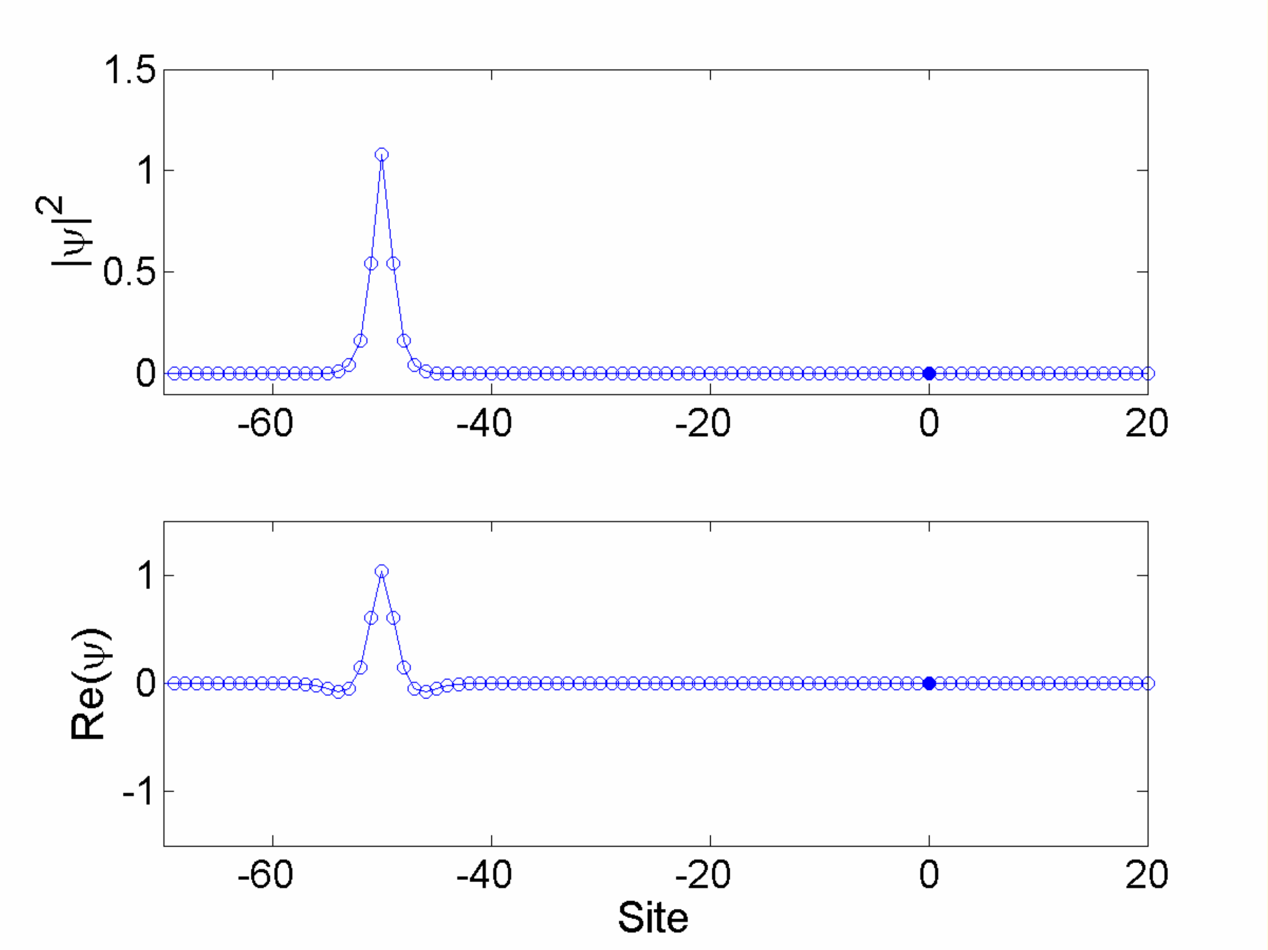
Trapping and reflection (II)

- Attractive impurity and strong enough. Some fraction of energy remains trapped and a considerable amount of it is reflected, remaining localized.
- The incident traveling structure has enough energy to excite the stationary mode centered at the impurity.
- The frequency of the remaining trapped mode is slightly lower than that the incident breather, so it has even smaller energy and power than the corresponding nonlinear mode with the frequency of incident soliton.



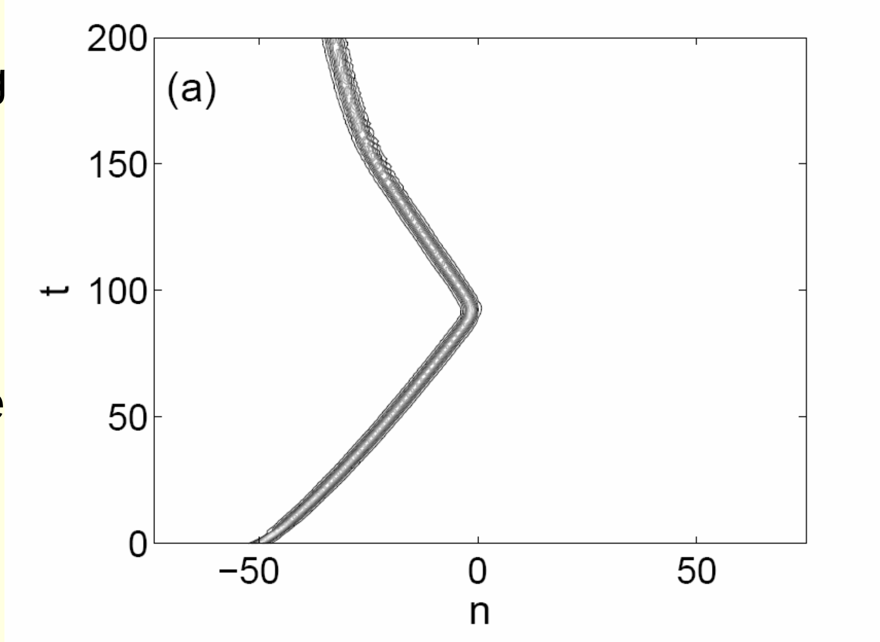
We have found that a **<u>necessary condition</u>** to trap energy and power by the impurity is the existence of a nonlinear localized mode centered at the impurity, with similar frequency, and energy and power smaller than that of the corresponding incident soliton.

III. Reflection with no trapping

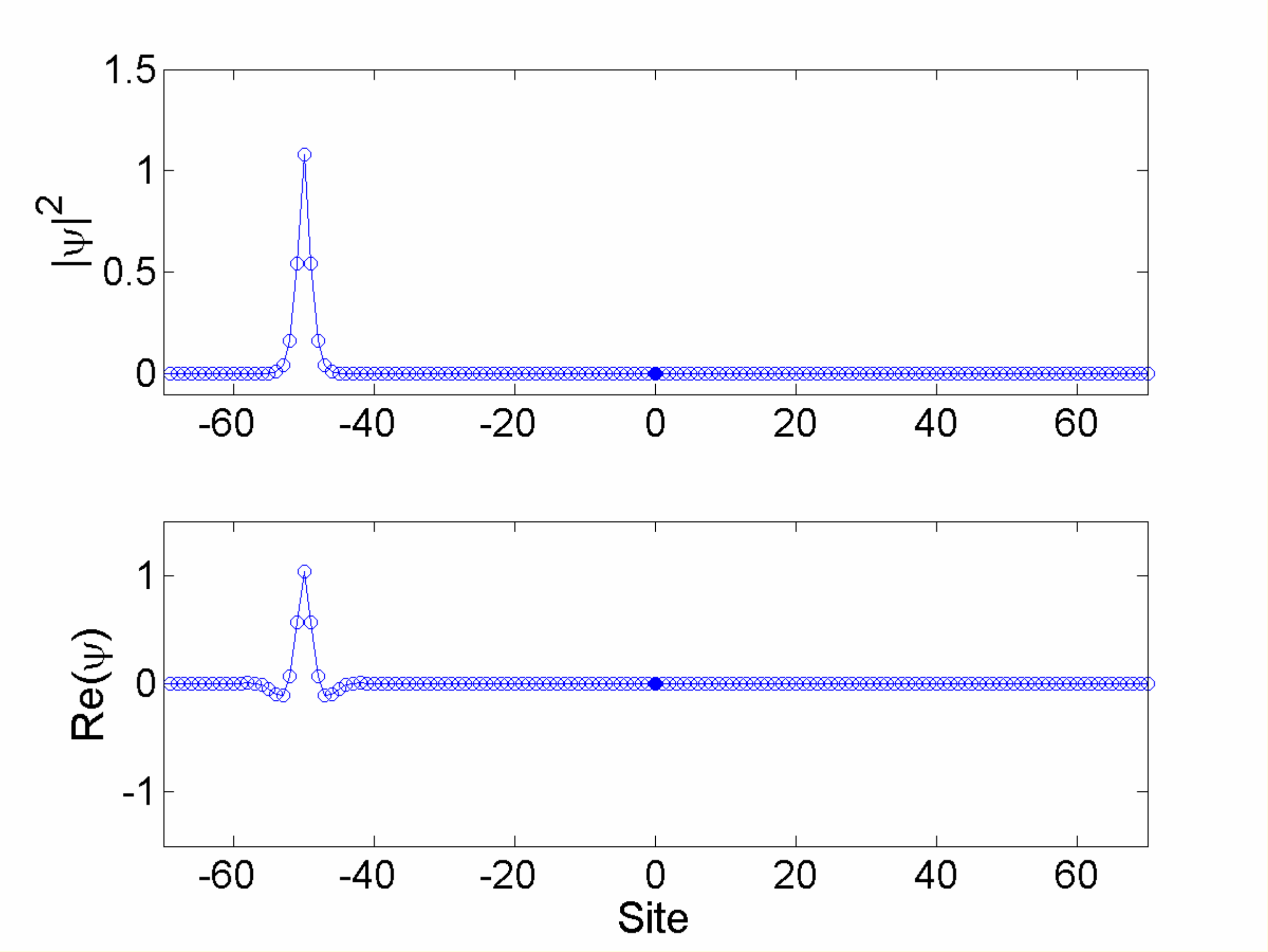


Reflection with no trapping (II)

- If the impurity is repulsive, and *q* small enough, neither trapping, nor transmission occur. Instead, all energy is reflected, and the traveling nonlinear excitation remains localized. The incident wave has no energy and power to excite the localized mode.
- If the impurity is attractive and strong enough, the frequency of the soliton is smaller than the corresponding to linear impurity mode, and all the energy is reflected. This is in accordance with the necessity of a nonlinear localized mode at the impurity site in order for the trapping to occur.

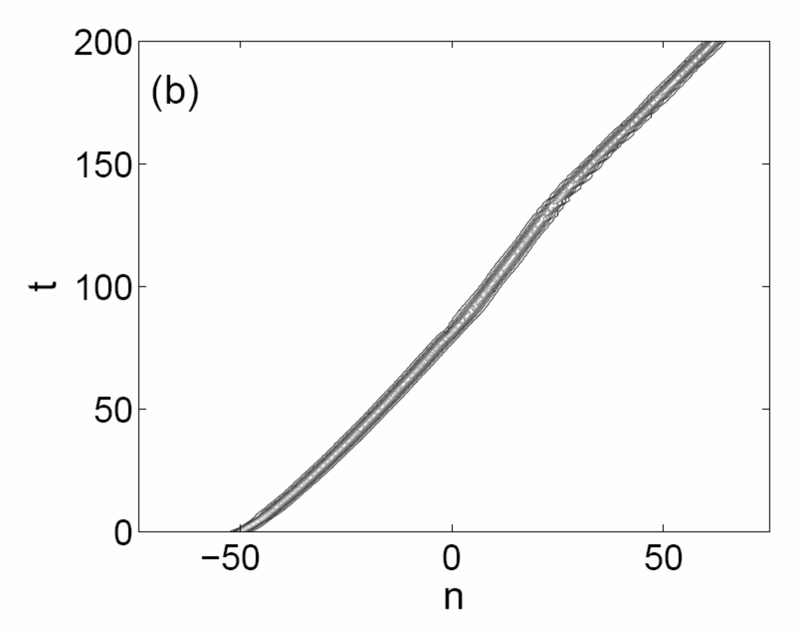


IV. Transmission with no trapping

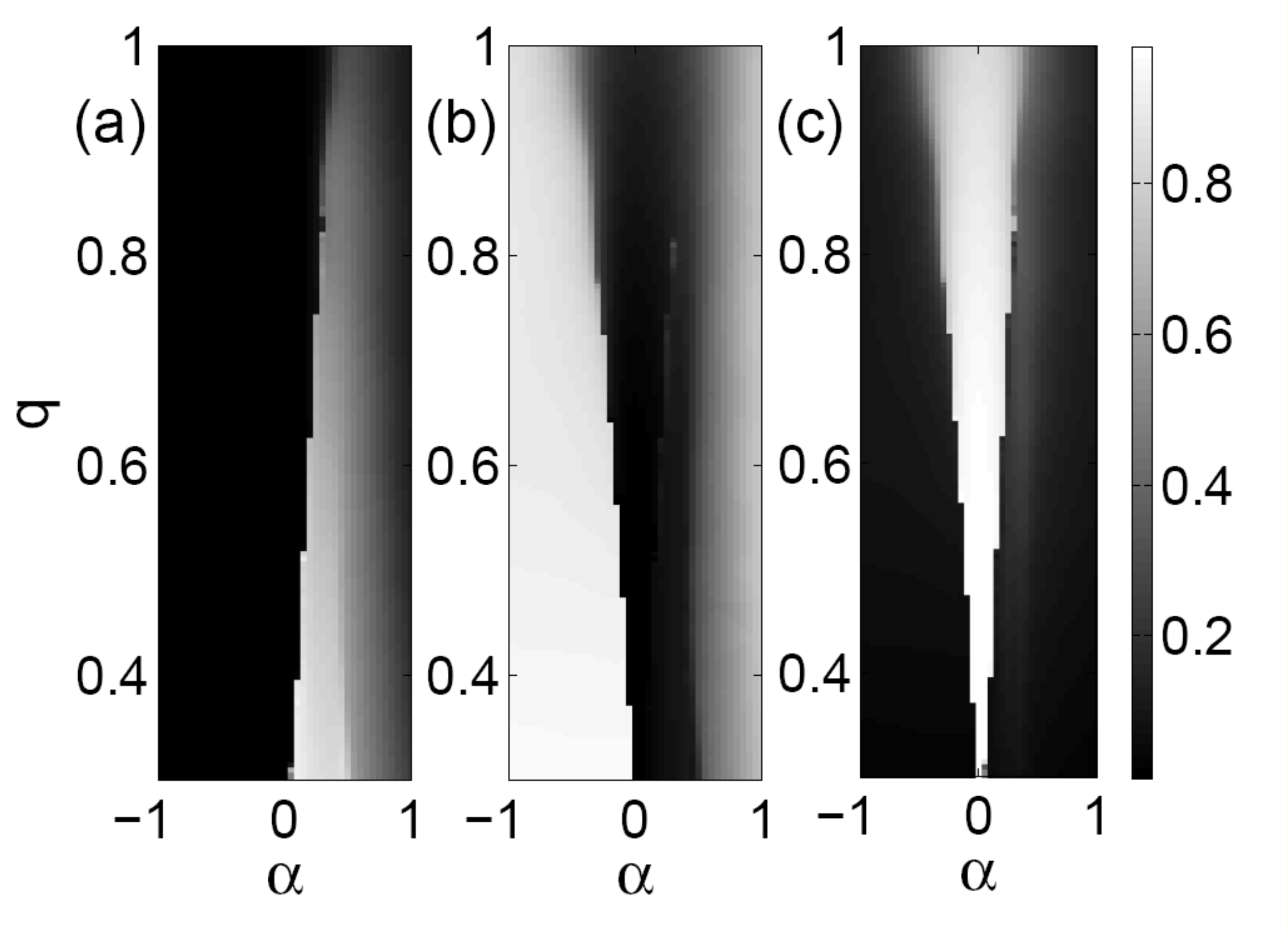


Transmission with no trapping (II)

- If $|\alpha|$ is small enough, and q high enough, transmission with no trapping occurs. There exists a critical value of $q=q_c>0$ that, if $q>q_c$, the incident soliton crosses through the impurity. The value of q_c grows with $|\alpha|$. In the case where $q<q_c$, if $\alpha<0$, reflection with no trapping occurs, while if $\alpha>0$, trapping with no reflection phenomenon takes place.

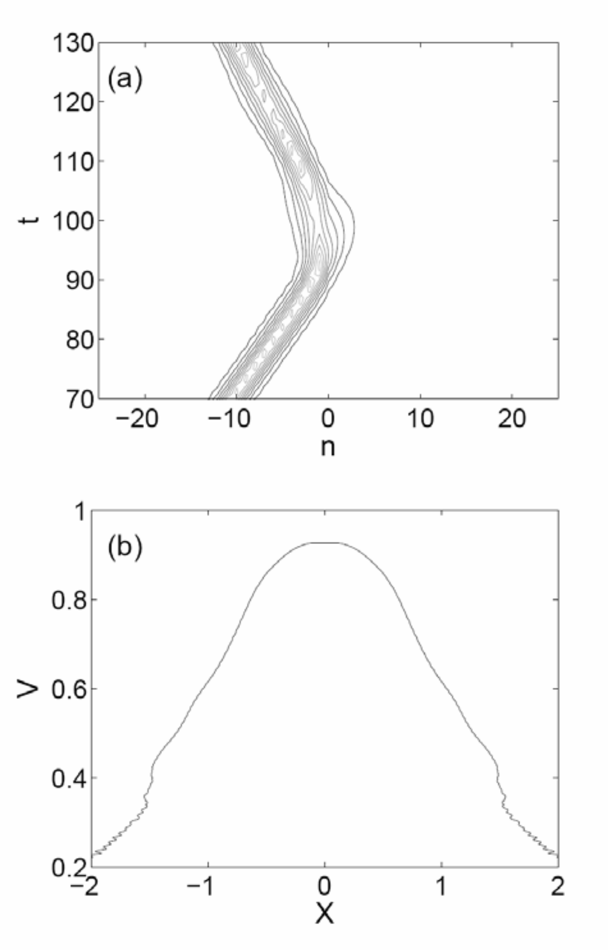


Power trapping, reflection and transmission coefficients



Soliton as a “quasiparticle”

- If parameter α is negative and small (in absolute value) enough. In this case, the solitary wave can be reflected or transmitted depending on its velocity. Also, when it is reflected, our numerical tests show that its velocity is similar to its incident velocity.
- We can consider the soliton as a `”quasiparticle'', and the effect of the impurity is similar to the effect of a potential barrier. We can determine this potential barrier for a given value of parameter α,
- If the parameter α is small enough, and positive (attractive), the solitary wave faces a potential “well'' and can be trapped if its translational energy is small or, if the translational energy is high enough, it may be transmitted, losing energy that remains trapped by the impurity, and decreasing its velocity.



Comparison with other related models

- Stationary solitons with a quintic nonlinear impurity. Bifurcation diagram of solitons close to impurity similar to the linear impurities case (P.G. Kevrekidis et al. PRE 67, 046604, 2003).

$$i\dot{u}_n + |u_n|^2 u_n + \epsilon(u_{n+1} + u_{n-1} - 2u_n) + \alpha\delta_{n,0}|u_n|^4 u_n = 0$$

- Small-amplitude solitons with either a linear and a nonlinear cubic impurity. Transmission, trapping and reflection (L. Morales-Molina and R.A. Vicencio, Opt. Lett. 31, 966, 2006).

$$i\dot{u}_n + (1 + \alpha\delta_{n,0})|u_n|^2 u_n + \epsilon(u_{n+1} + u_{n-1}) = 0$$

- Klein-Gordon chains. Moving breathers qualitatively equivalent to DNLS solitons (J. Cuevas et al Theor. Math. Phys. 137, 1406, 2003).

$$\ddot{u}_n + V_n'(u_n) + C(2u_n - u_{n+1} - u_{n-1}) = 0$$

Future challenges

- Consideration of saturable nonlinearities which can describe nonlinear waveguides made of photorefractive materials. This kind of nonlinearity may allows the possibility of moving solitons may enhance the mobility of solitary waves in isotropic two-dimensional lattices whereas moving solitons only can take place in anisotropic 2D lattices for cubic lattices.
- Inclusion of two or more local inhomogeneities and the examination of the interplay between them

References

- P. G. Kevrekidis. The Discrete Nonlinear Schrödinger Equation: Mathematical Analysis, Numerical Computations and Physical Perspectives. Springer, to appear.
- F Palmero, R Carretero-González, J Cuevas, PG Kevrekidis and W Królikowski. PRE, 77, 036614, 2008.