Dark breathers
A study of their existence and stability

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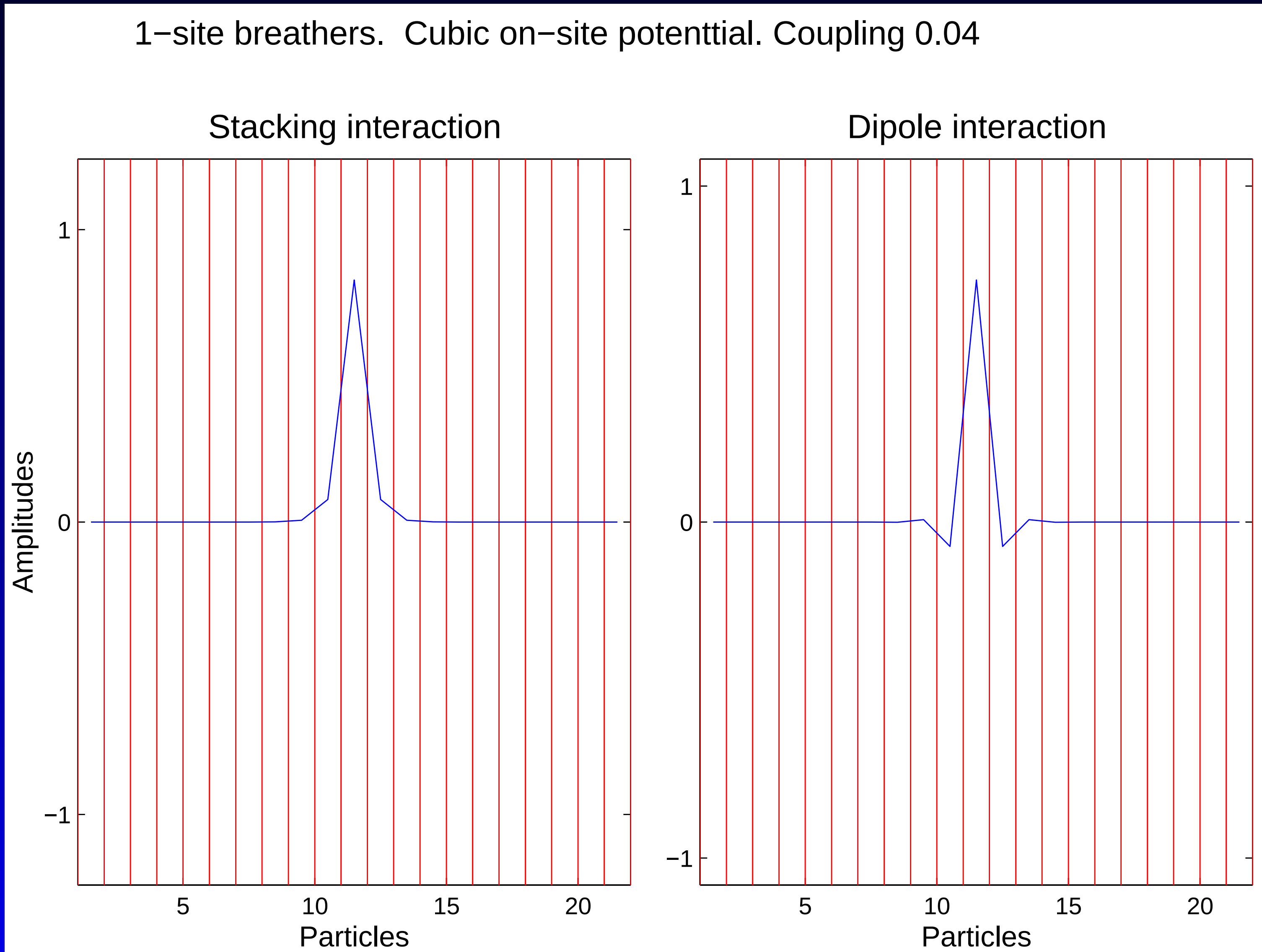
Bright and dark breathers

- What is a discrete (bright) breather?
  - Localized, periodic oscillations in a discrete system.

$$\max|u_n(t)| < ke^{-b|n|}$$

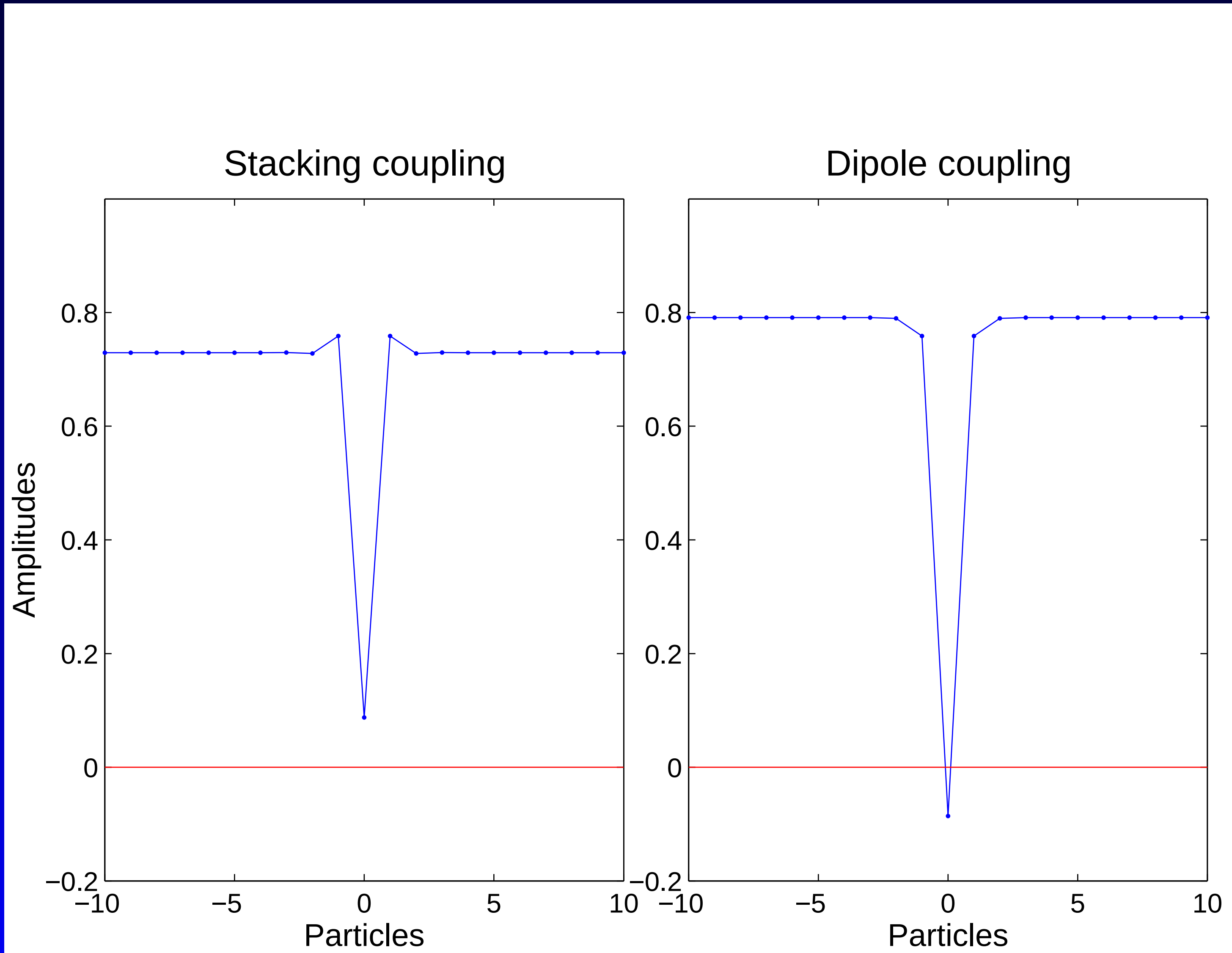
- What is a discrete dark breather?
  - One oscillator with small amplitude
  - A bakground of excited oscillators
- Do exist dark breathers?
- Are they stable?
- Relationship with dark solitons?

Bright soft breathers



Dark cubic breathers

Coupling $\varepsilon = 0.023$



Hamiltonian

- Hamiltonian

$$H = \sum_n (\frac{1}{2}\,\dot{u}_n^2 + V(u_n)) + \varepsilon\, W(u)$$

$$u = (u_1, u_2, \ldots, u_n)$$

- Cubic on–site potential and stacking, harmonic coupling

$$V(u_n) = \frac{1}{2}u_n^2 - \frac{1}{3}u_n^3$$

$$W(u) = \frac{1}{2}\sum_n (u_{n+1} - u_n)^2$$

Equations and codes

- Dynamical equations

$$\ddot{u}_n + V'(u_n) + \varepsilon(2u_n - u_{n-1} - u_{n+1}) = 0$$

- Anticontinuous limit $\varepsilon = 0$
  - Time–symmetric solutions: $u_n(-t) = u_n(t)$
  - With given frequency ω_b
- Two oscillator states
  - At rest: $u_n = 0$
    code: $\sigma_n = 0$
  - Excited: $u_n + V'(u_n) = 0$
    code $\sigma_n = +1 \quad \text{if } \dot{u}_n(0) < 0$
    code $\sigma_n = -1 \quad \text{if } \dot{u}_n(0) > 0$

Different breathers and codes

- One-site breather

$$\sigma = \{0, ..., 0, 1, 0, ..., 0\}$$

- Symmetric double breather

$$\sigma = \{0, ..., 0, 1, 1, 0, ..., 0\}$$

- Phonobreather

$$\sigma = \{1, 1, ..., 1, 1\}$$

- One–site, dark breather

$$\sigma = \{1, ..., 1, 0, 1, ..., 1\}$$

Mackay and Aubry theorem

- Hypotheses:
  - The excited oscillators are nonlinear at ω_b

$$\left(\frac{\partial \omega}{\partial I}\right)_{\omega=\omega_b} \neq 0 \text{ for } I = \sum_n \int \dot{\xi}_n \, d\xi_n$$

  - No harmonic coincides with the linear frequency
    $p\,\omega_b \neq \omega_0 \equiv \sqrt{V''(0)} \quad \forall p \in \mathbb{N}$
- Result: There exist solutions for $\varepsilon \neq 0$, $|\varepsilon| < \varepsilon_c$
  - For any code
  - Exponentially localized (one–site breather)
  - Stable for $0 < \varepsilon < \varepsilon'_c$ (one–site breather)

Conclusion:

Dark breathers exist

- **Up to what coupling?**
- **Are they stable?**

Stability analysis

- Perturbations $\xi(t)$
  $\tilde{u}_n(t) = u_n(t) + \xi_n(t)$ with $\xi_n(t) \in \mathrm{C}^2$
- Newton operator

$$(\mathcal{N}(u(t),\varepsilon)\cdot\xi\,)_n \equiv$$
$$\ddot{\xi}_n \; + \; V''(u_n)\,\xi_n + \varepsilon(2\xi_n - \xi_{n-1} - \xi_{n+1})$$

- $V''(u(t))$ and $\mathcal{N}$ with period $T_b = 2\pi/\omega_b$
- Eigenvalue equation

$$\mathcal{N}(u(t),\varepsilon)\cdot\xi \; = \; E\,\xi$$

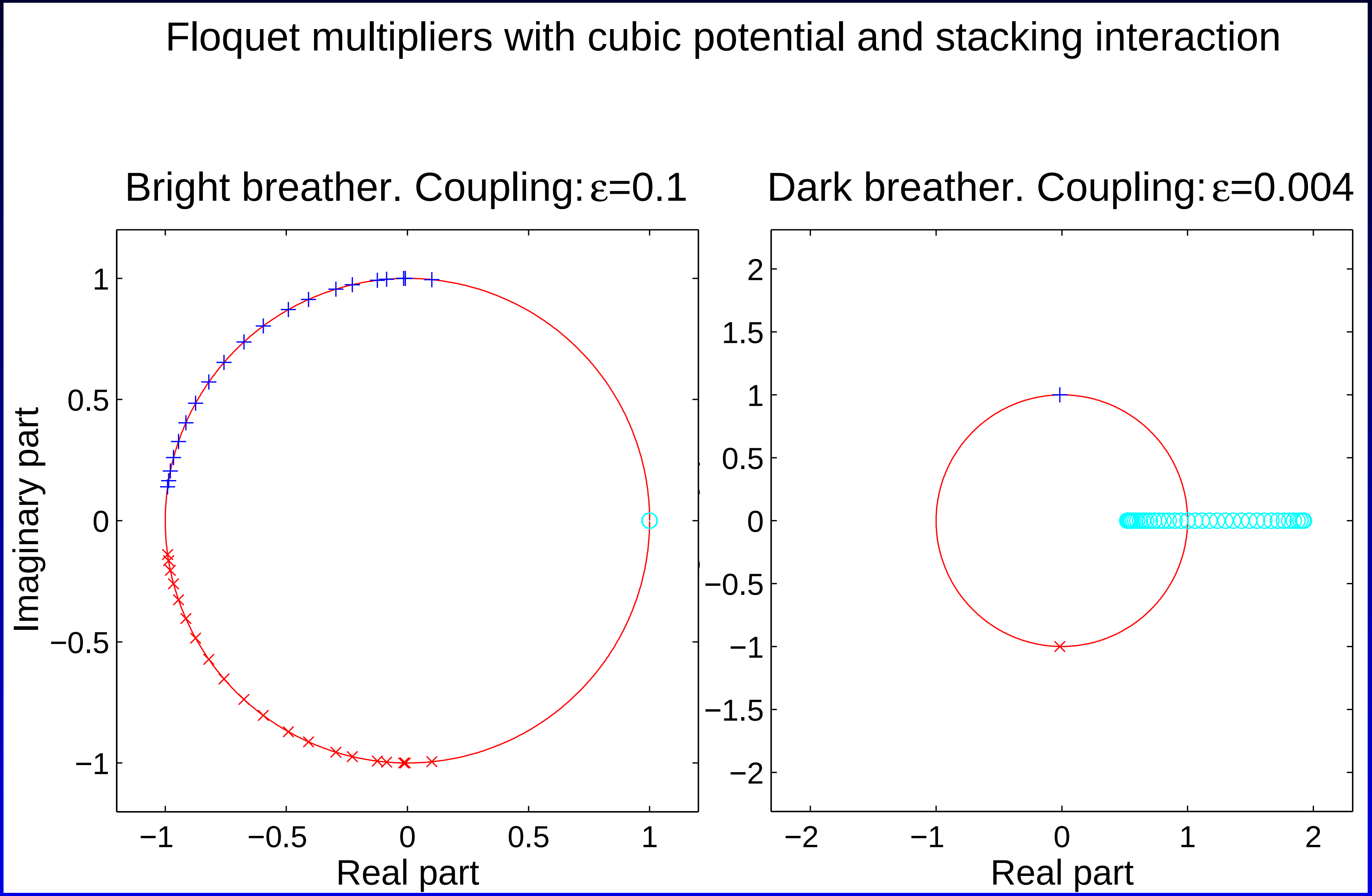
- Perturbations: eigenfunctions of $\mathcal{N}$ with eigenvalue $E = 0$

Floquet operator

$$\begin{pmatrix} \{\xi_n(T_b)\} \\ \{\dot{\xi}_n(T_b)\} \end{pmatrix} = \mathcal{F}_0 \begin{pmatrix} \{\xi_n(0)\} \\ \{\dot{\xi}_n(0)\} \end{pmatrix}$$

- Floquet multipliers λ_j= eigenvalues of $\mathcal{F}_0$
- Stability in dissipative systems: $|\lambda_j| < 1$
- Properties in real Hamiltonian systems:
  - $\mathcal{F}_0$ is real:
    $\lambda_j \in \mathrm{spec}(\mathcal{F}_0) \Rightarrow \lambda_j^* \in \mathrm{spec}(\mathcal{F}_0)$
  - $\mathcal{F}_0$ is symplectic:
    $\lambda_j \in \mathrm{spec}(\mathcal{F}_0) \Rightarrow 1/\lambda_j \in \mathrm{spec}(\mathcal{F}_0)$
  - Stability: $|\lambda_j| = 1$ or $\lambda_j = e^{\mathrm{i}\theta_j}$, θ_j real

Some Floquet multipliers



Multipliers at zero coupling

- General:
  Double $\lambda = 1 + 0\,\mathrm{i} \Leftarrow \mathcal{N} \cdot \dot{u}(t) = 0$
- At zero coupling:
  - Oscillators at rest: $\ddot{\xi}_n + \omega_0^2\,\xi_n = 0$

    $$\lambda_\pm = e^{\pm\,\mathrm{i}\,\omega_0\,T_b} = e^{\pm\,\mathrm{i}\,2\,\pi\,\omega_0/\omega_b} \neq 1$$

  - Excited oscillators: $\ddot{\xi}_n + V''(u_n(t))\,\xi_n = 0$
    Double $\lambda = 1 \Leftarrow \mathcal{N}_n \cdot \dot{u}_n(t) = 0$

Possible instability bifurcations

Collisions of multipliers:

- Harmonic bifurcations:
  2 multipliers at $\lambda = 1$ or $\theta = 0$
- Subharmonic bifurcations:
  2 multipliers at $\lambda = -1$ or $\theta = \pm\pi$
- Krein crunches (oscillatory instabilities)
  4 multipliers at $\lambda_{\pm} = e^{\pm \mathrm{i}\theta}$

Krein signature restrictions

Krein signature: $k(\theta) = \text{sign}(\mathrm{i}\,(\xi \cdot \dot{\xi}^* - \xi^* \cdot \dot{\xi}))$
$\xi(t)$ eigenfunction with multiplier $e^{\mathrm{i}\theta}$

- Rest oscillators at zero coupling

$$k(\pm 2\pi\omega_0/\omega_b) = \pm 1$$

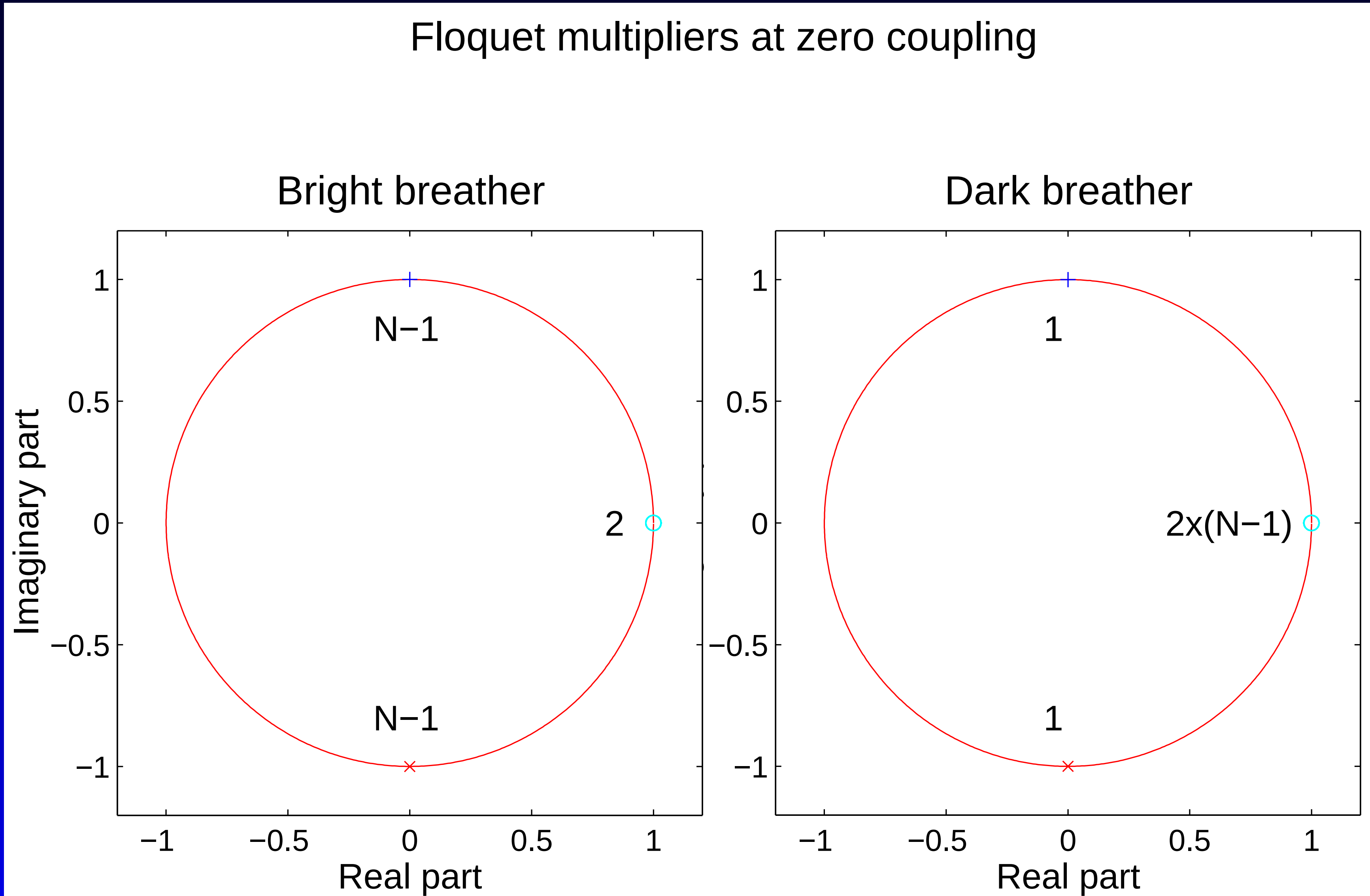
- Excited oscillators at zero coupling

$$k = 0 \quad \Leftarrow \quad \xi(t) \text{ is real}$$

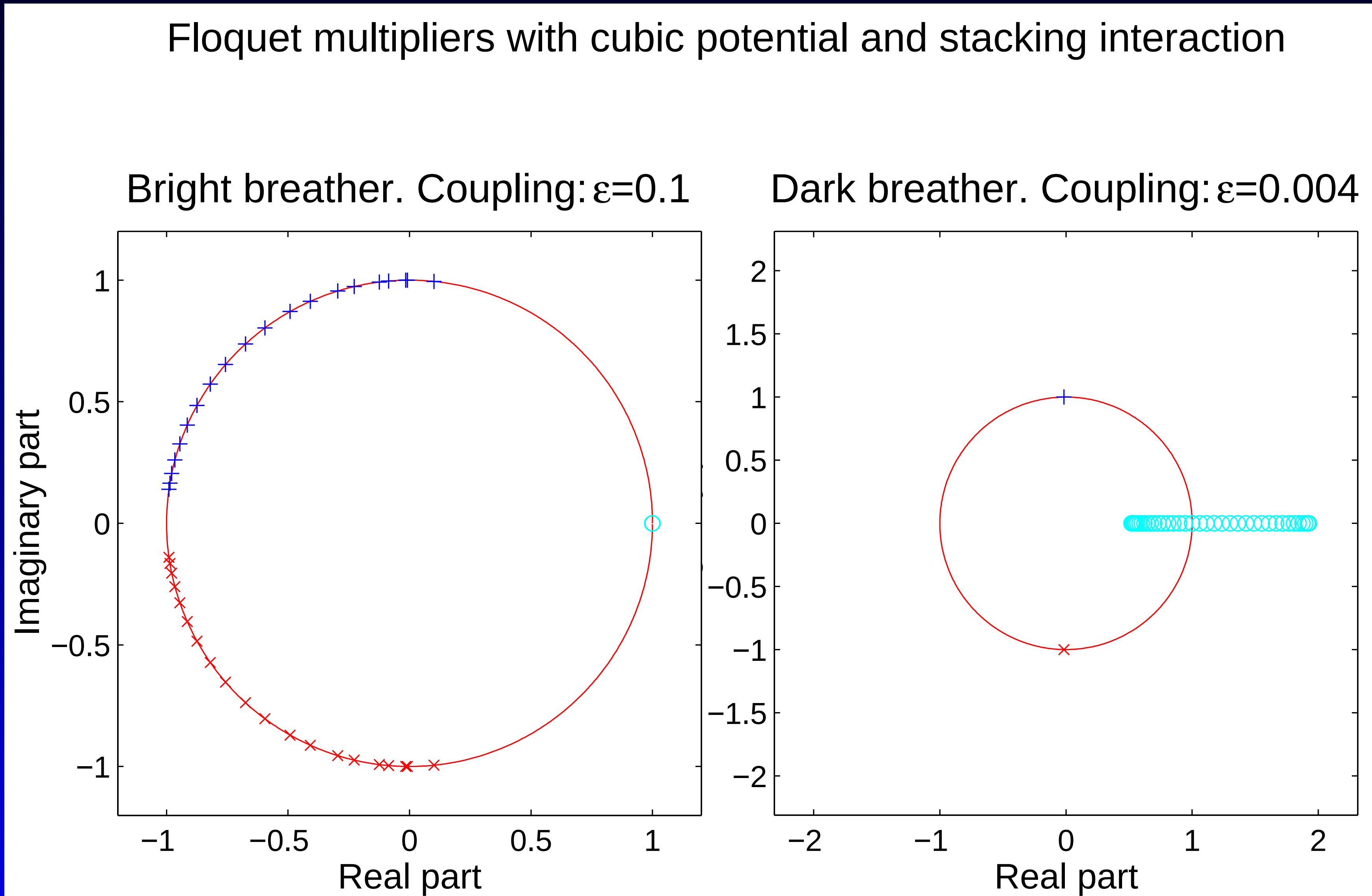
Krein condition:

It is not possible a bifurcation involving two multipliers with Krein signatures of the same sign.

Multipliers at zero coupling



Multipliers with coupling



Aubry's band theory

- Eigenvalue equation for the Newton operator

$$\mathcal{N}(u(t),\varepsilon)\cdot\xi(t) = E\,\xi(t)$$

- Floquet operator $\mathcal{F}_E$

$$\begin{pmatrix} \{\xi_n(T_b)\} \\ \{\dot{\xi}_n(T_b)\} \end{pmatrix} = \mathcal{F}_E \begin{pmatrix} \{\xi_n(0)\} \\ \{\dot{\xi}_n(0)\} \end{pmatrix}$$

- $\mathcal{N}$ commutes with $\mathcal{P}$: $\quad \mathcal{P}\cdot\xi(t) = \xi(t+T_b)$
- The eigenfunctions are Bloch functions $\xi(t) = e^{\mathrm{i}\,\theta\,t/T_b}\,\chi(t), \quad ; \quad \chi(t+T_b) = \chi(t)$
- Bloch Floquet multiplier $e^{\mathrm{i}\,\theta}$, $\theta\,\in\,\mathbb{C}$

Band structure

- Set of points $(\theta_1, E), (\theta_2, E), \cdots, (\theta_{2N}, E)$, when θ_n is real
- Properties
  - symmetric with respect to θ:

$$E(-\theta) = E(\theta)$$

  - Reduced to the first Brillouin zone:

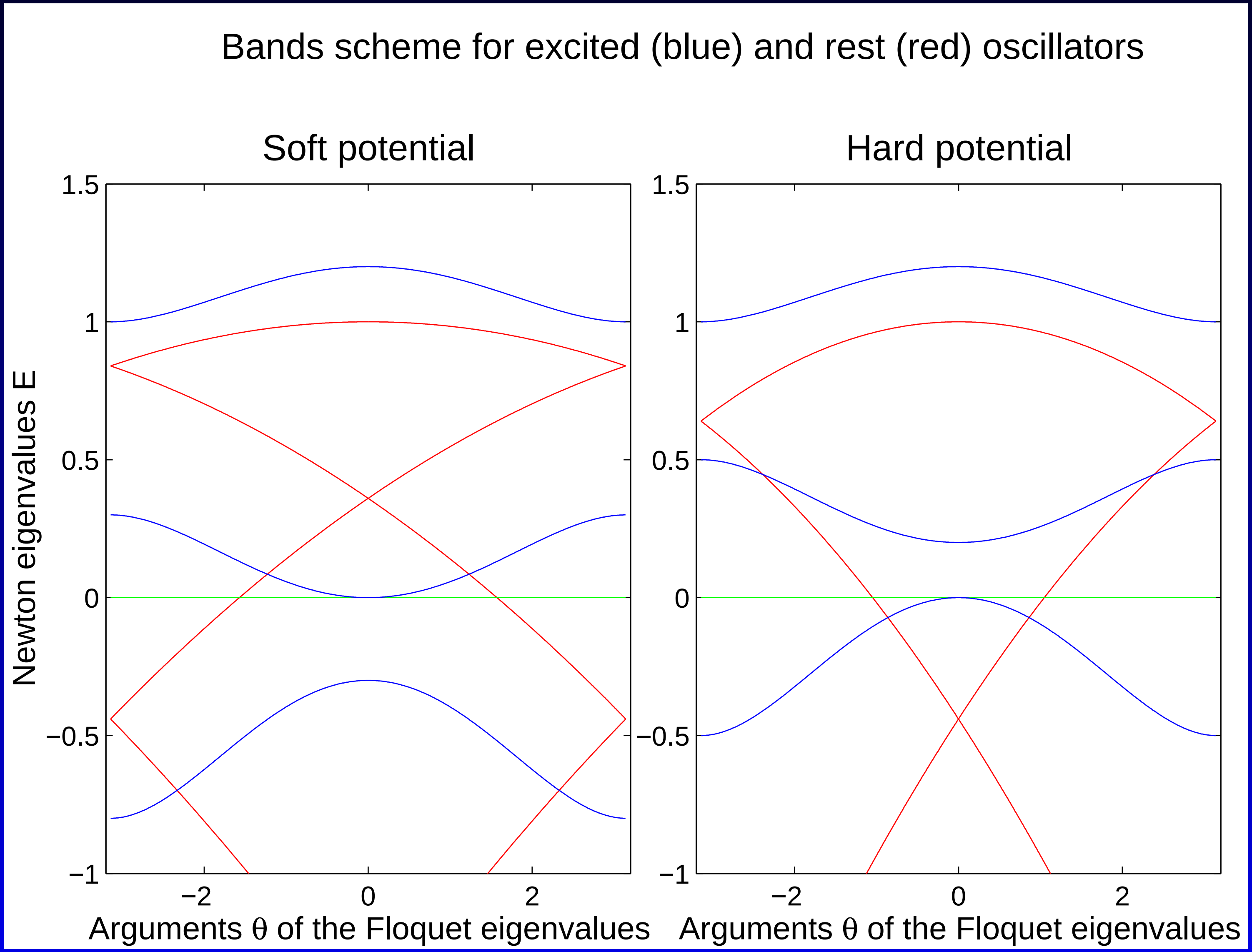
$$\theta \in [-\pi, \pi]$$

- Band formulation of breather stability: *A breather is stable if there are $2N$ band intersections with the axis $E = 0$.*

Bands at zero coupling

- Rest oscillators: $\ddot{\xi}_n + \omega_0^2\,\xi_n = E\,\xi_n$
  - Solutions: $\xi_n^{\pm}(t) = e^{\pm\,\mathrm{i}\,\sqrt{\omega_0^2 - E}\;t}$
  - Floquet multipliers: $e^{\pm\,\mathrm{i}\,\sqrt{\omega_0^2 - E}}\;T_b$
  - Floquet arguments:
    $$\theta = \pm\sqrt{\omega_0^2 - E}\;T_b = \pm\sqrt{w_0^2 - E}\;\frac{2\pi}{\omega_b}$$
  - Bands: $E = \omega_0^2\, - \,\omega_b^2\left(\frac{\theta}{2\,\pi}\right)^2$
    reduced at $\theta \in [-\pi, \pi]$
- Excited oscillators
  - Deformation of the rest bands
  - Bands are tangent to $E = 0$ at $\theta = 0$ ($\lambda = 1$)

One oscillator bands



Band instability bifurcations

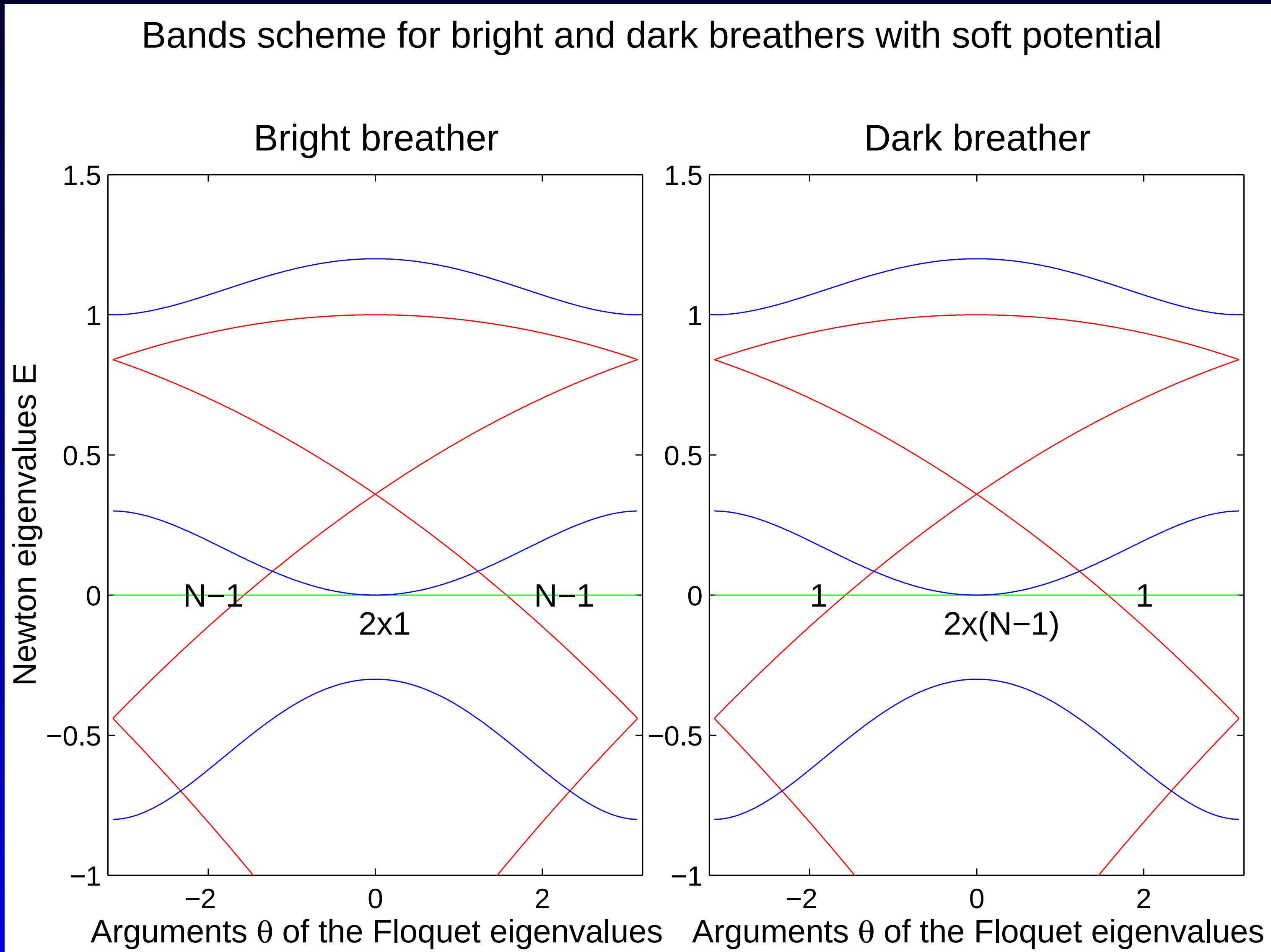
Bands loose intersection points with the $E = 0$ axis

- Harmonic bifurcations:
  A band tangent at $\theta = 0$ separates
- Subharmonic bifurcations:
  A band tangent at $\theta = \pm\pi$ separates
- Krein crunches (oscillatory instabilities)
  A band tangent at $\pm\theta \neq 0\ ;\theta \neq \pm\pi$ separates

Band interpretation of the Krein signature

- $k(\theta) = -\,\mathrm{sign}\left(\left(\frac{dE}{d\theta}\right)_{E=0}\right)$
- *Eigenvalues with the same sign of k belong to different bands and cannot bifurcate.*

Bands scheme at zero coupling



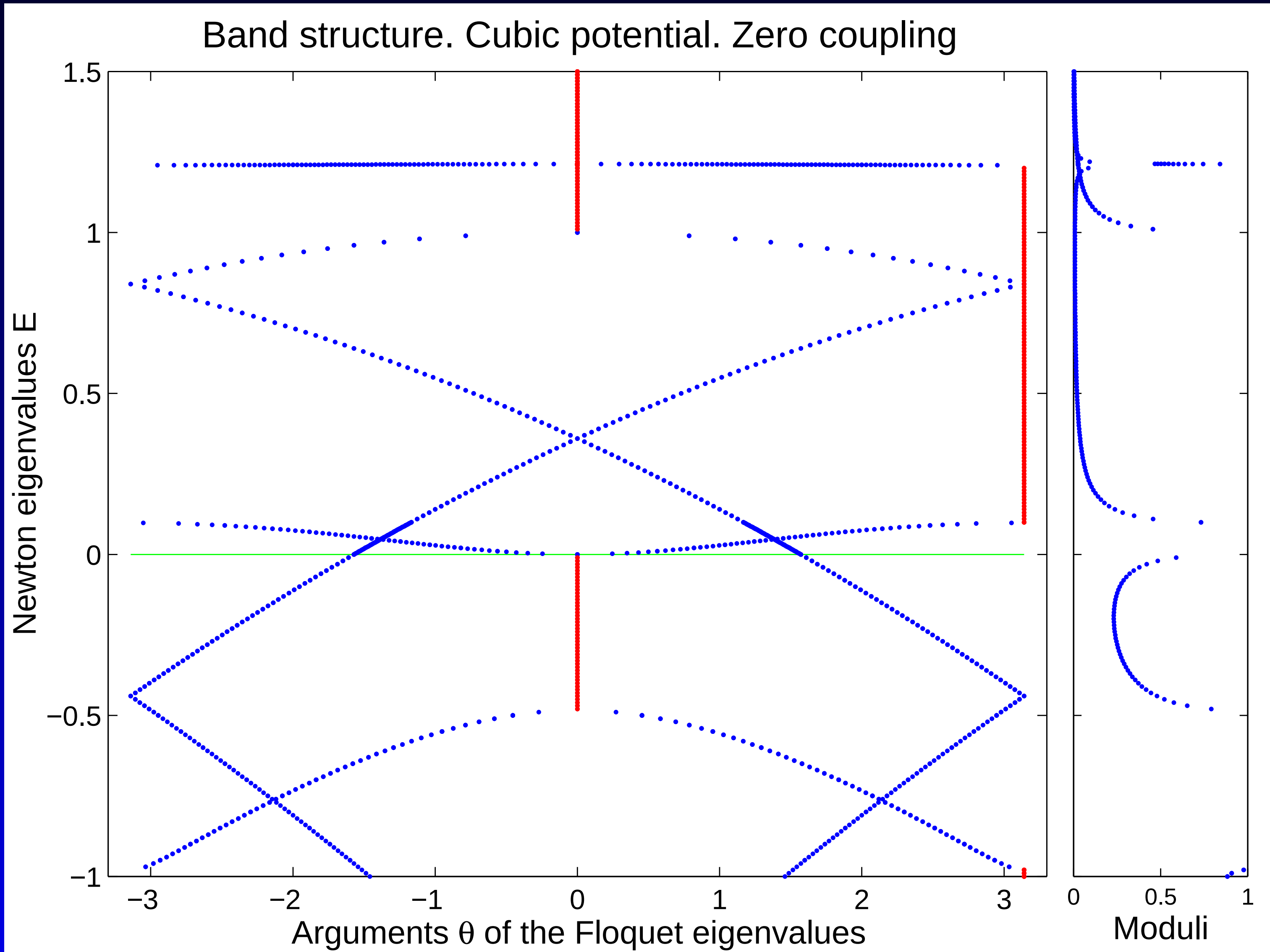
red: rest oscillators ; blue: excited oscillators

Bands movement

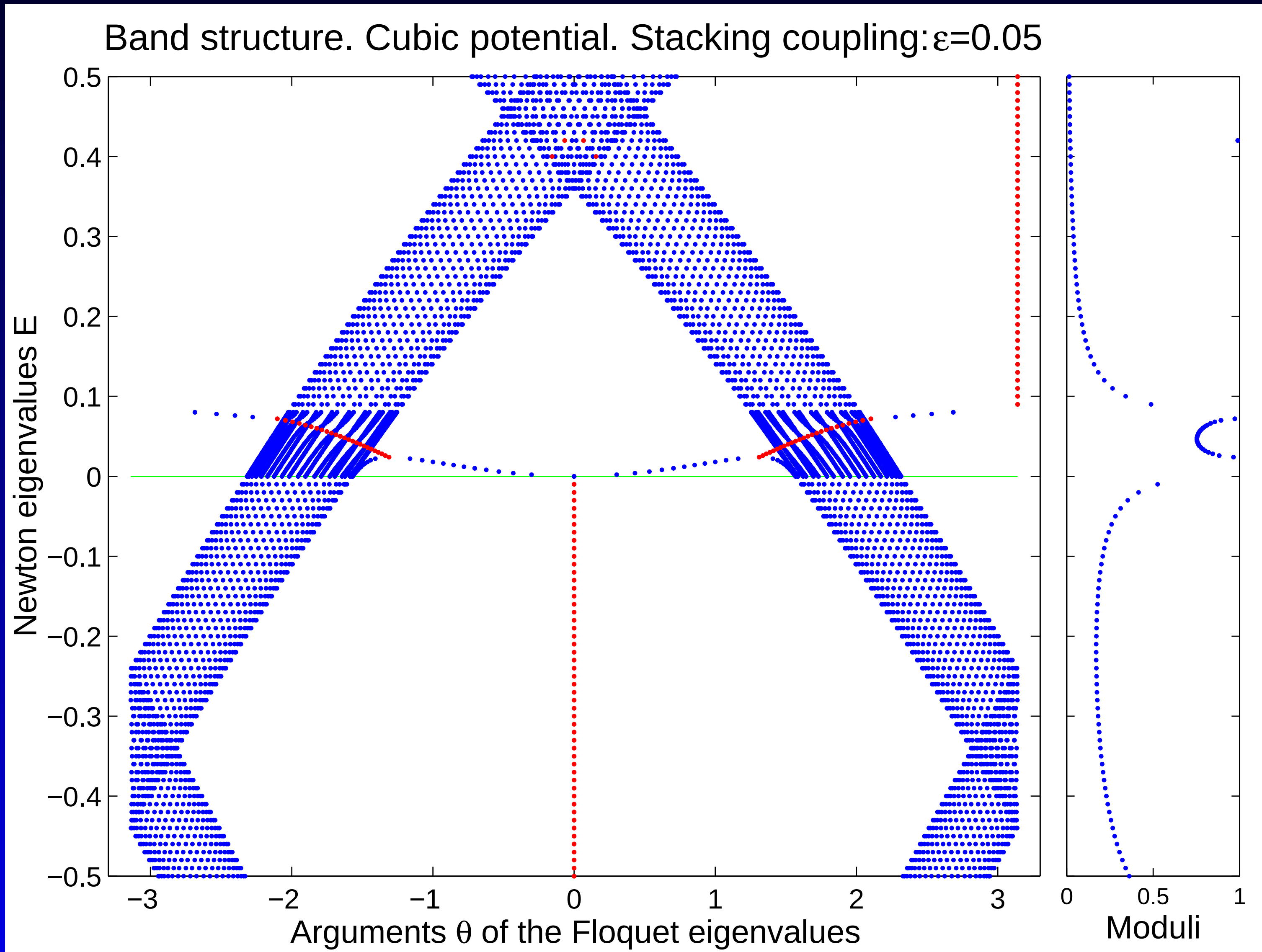
- With stacking coupling the bands move upwards, and dark breathers are unstable
- How can the bands move downwards?
  - Changing the sign of the coupling: $\ddot{u}_n + V'(u_n) - \varepsilon\,(2\,u_n - u_{n+1} - u_{n-1}) = 0$
  - Equivalently, with dipole–dipole interaction: $\ddot{u}_n + V'(u_n) + \varepsilon\,(u_{n+1} + u_{n-1}) = 0$

$$H = \sum_n \left(\tfrac{1}{2}\dot{u}_n^2 + V(u_n) + \frac{1}{2}\varepsilon\,(u_n\,u_{n+1} + u_{n-1}\,u_n)\right)$$

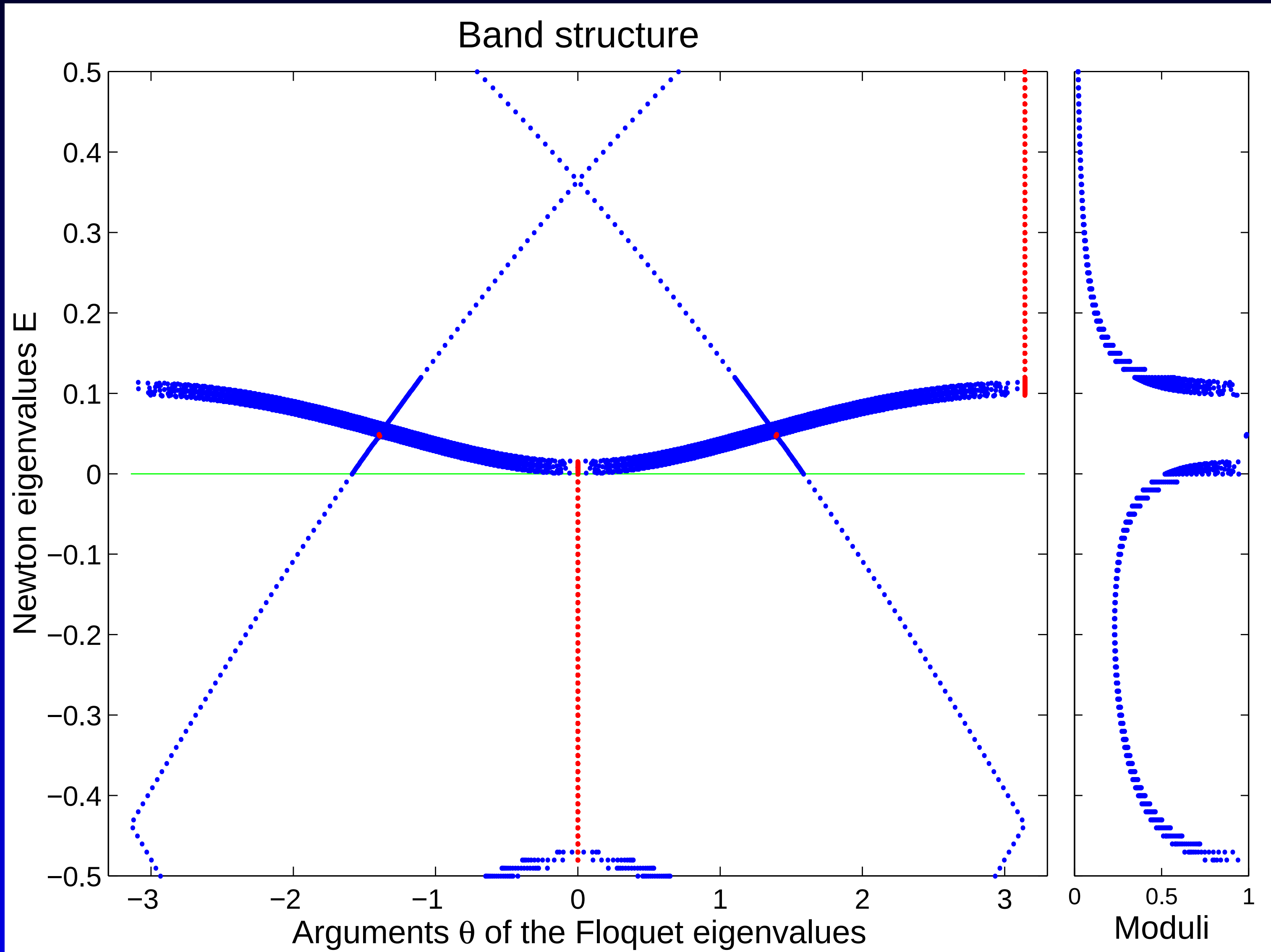
Calculated bands



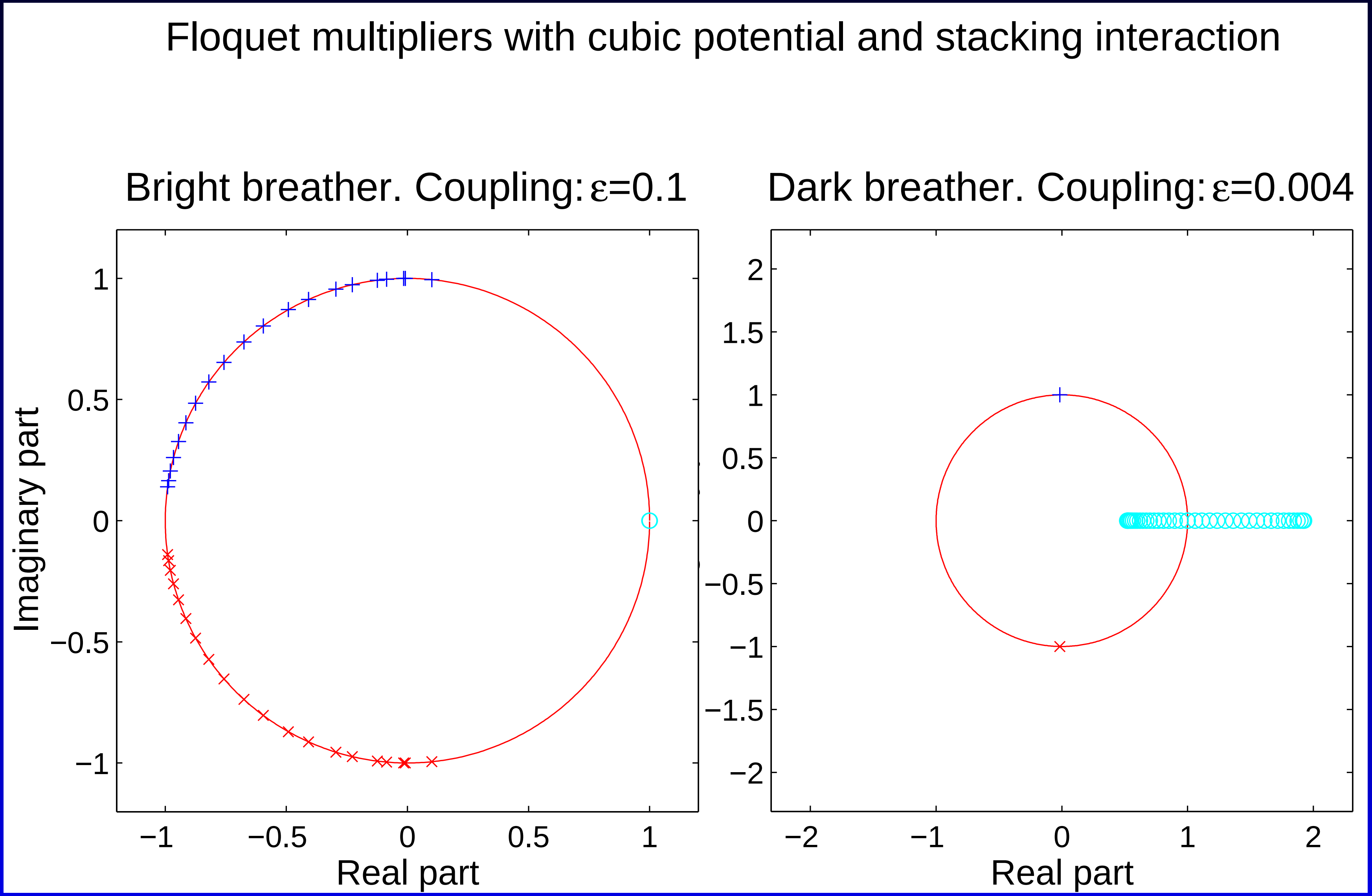
Bright breather bands



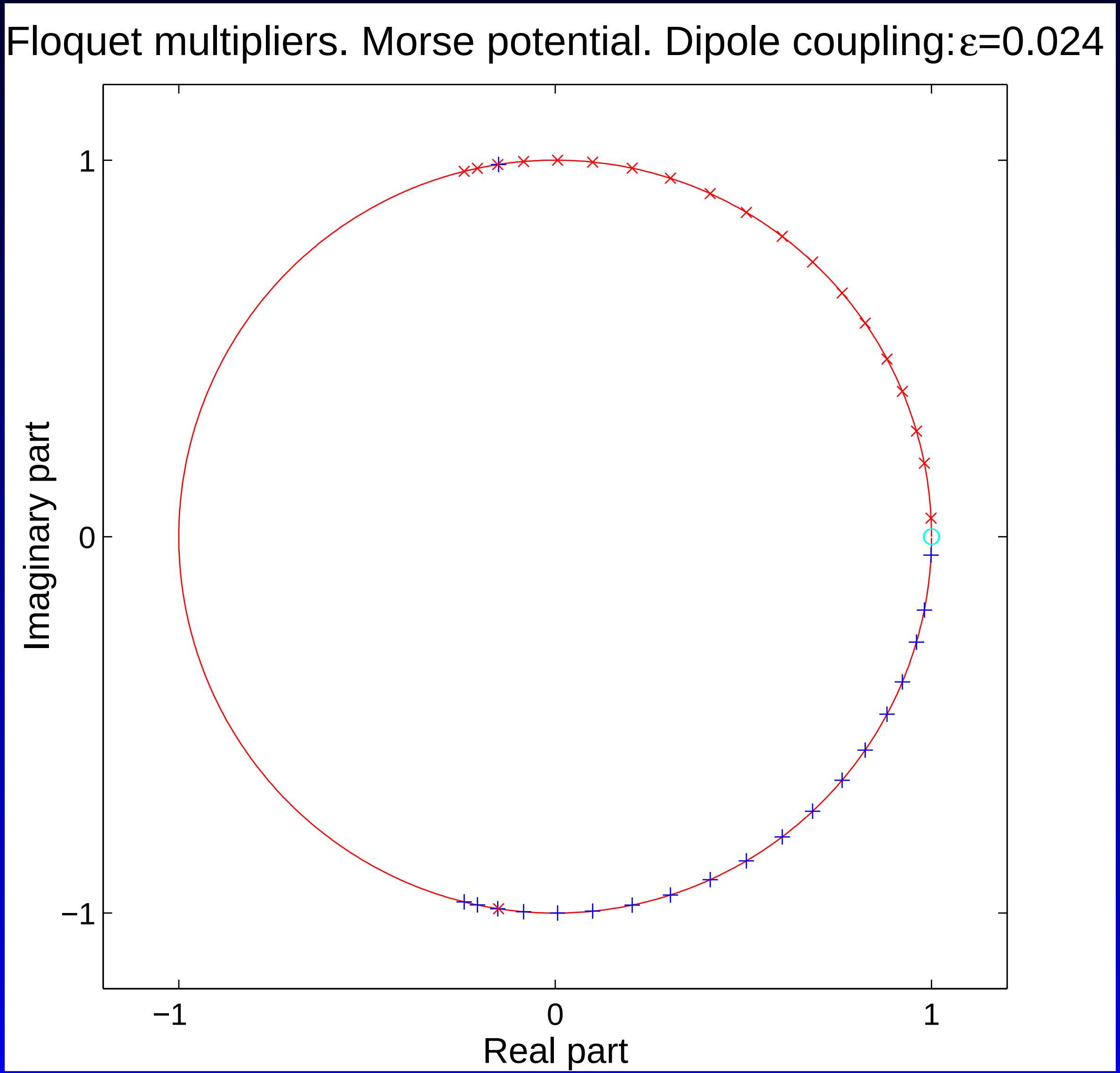
Dark breather bands



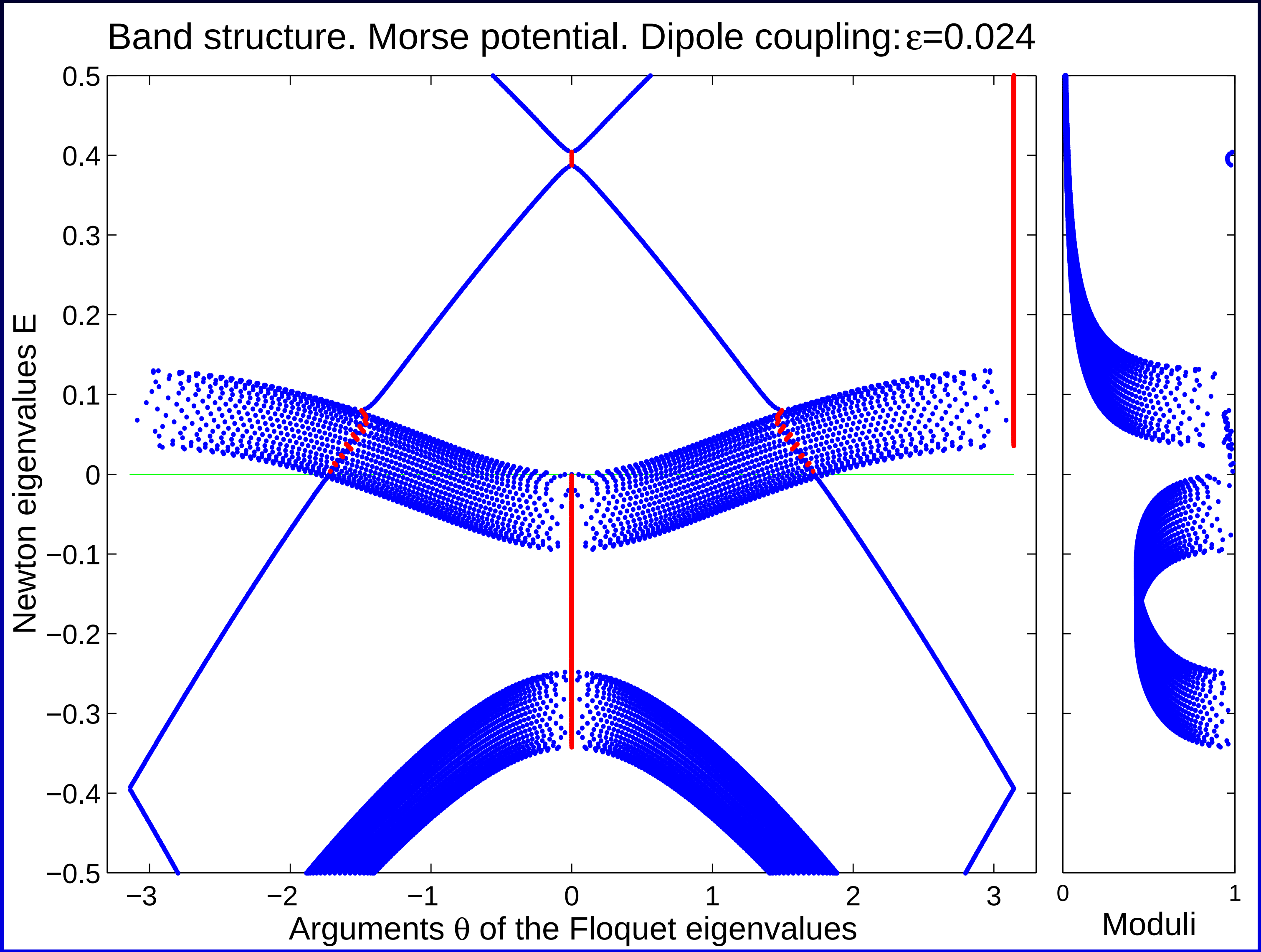
Corresponding multipliers



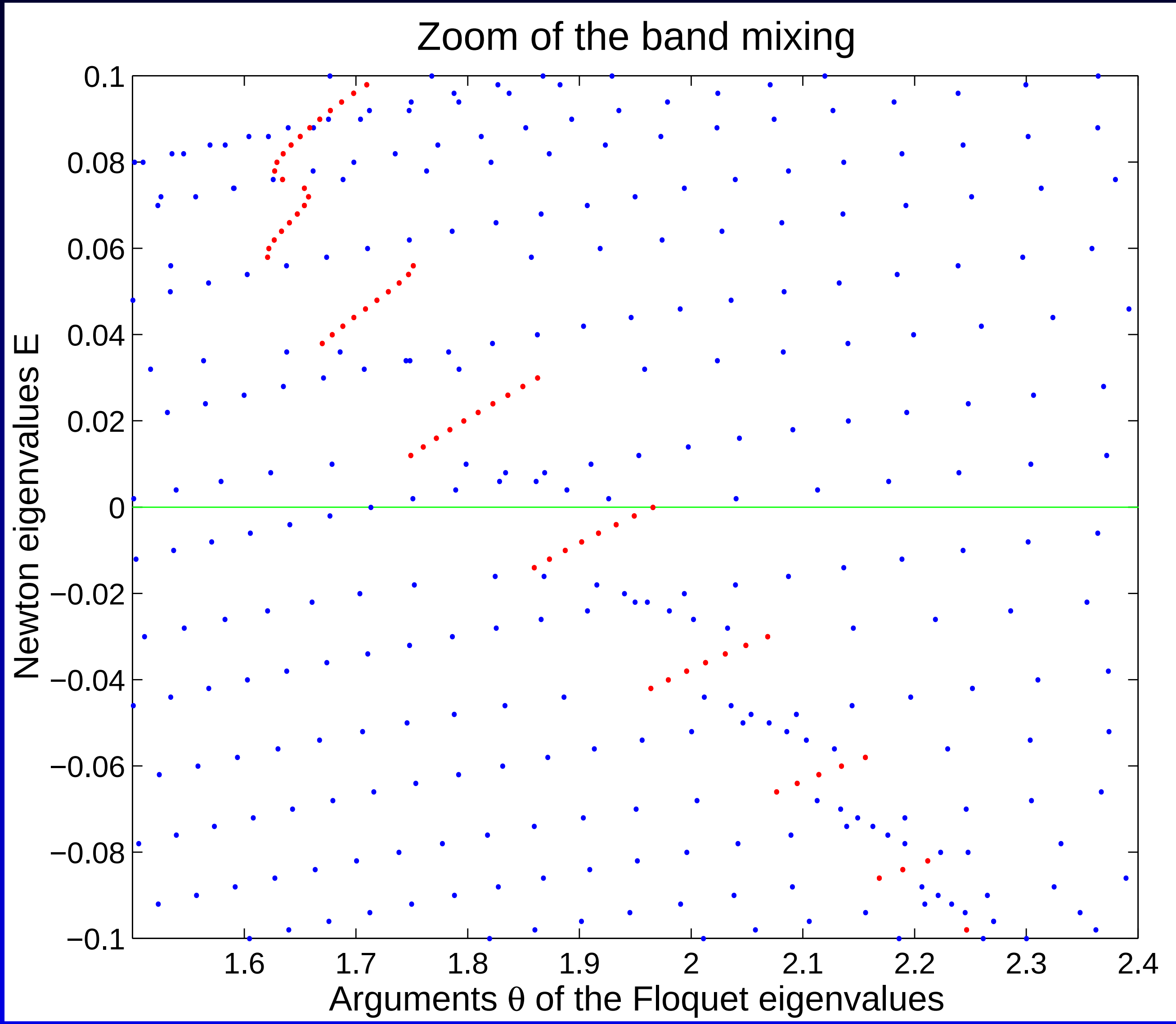
Stable dark breather



With Morse potential

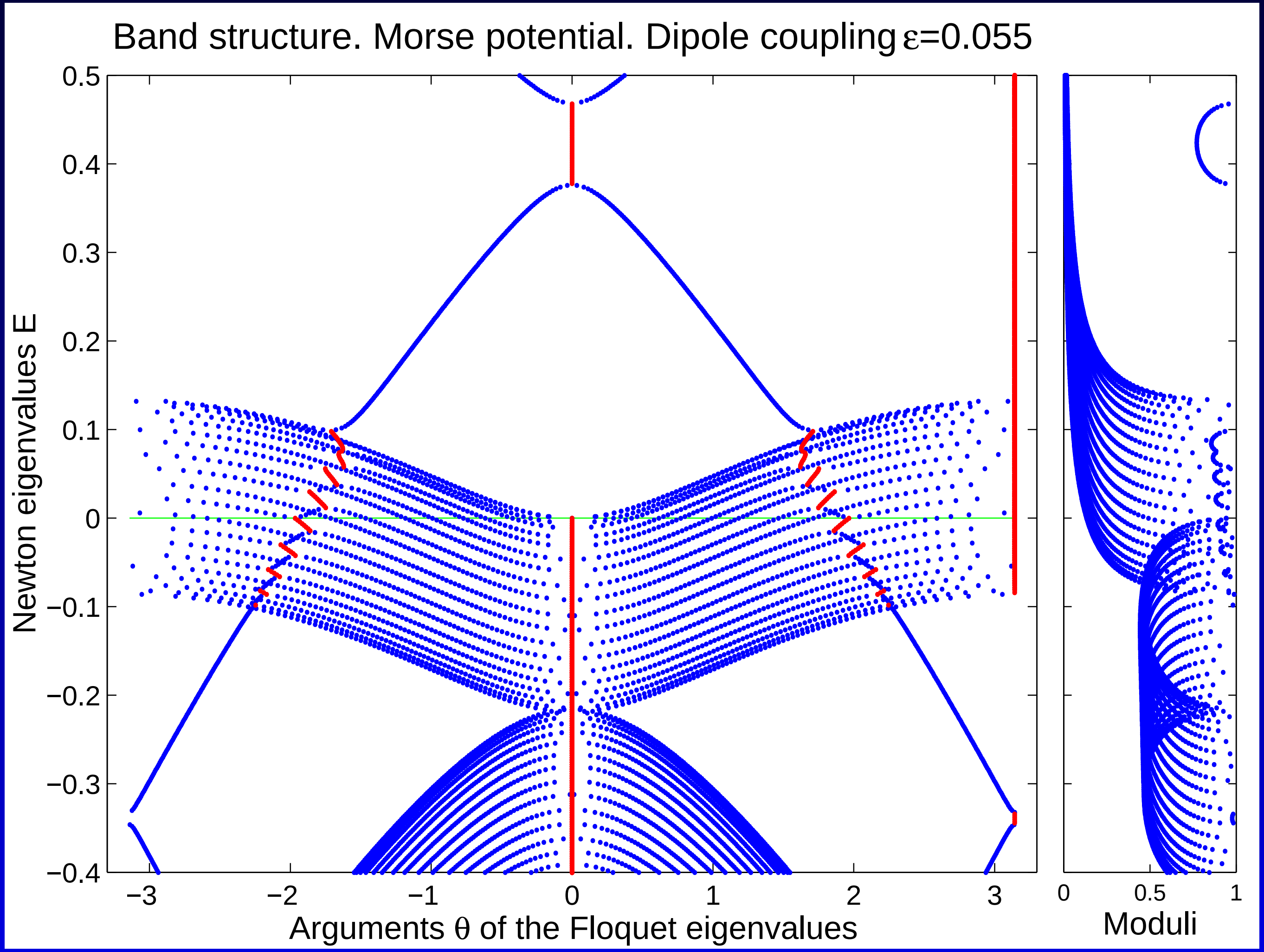


Band mixing zoom



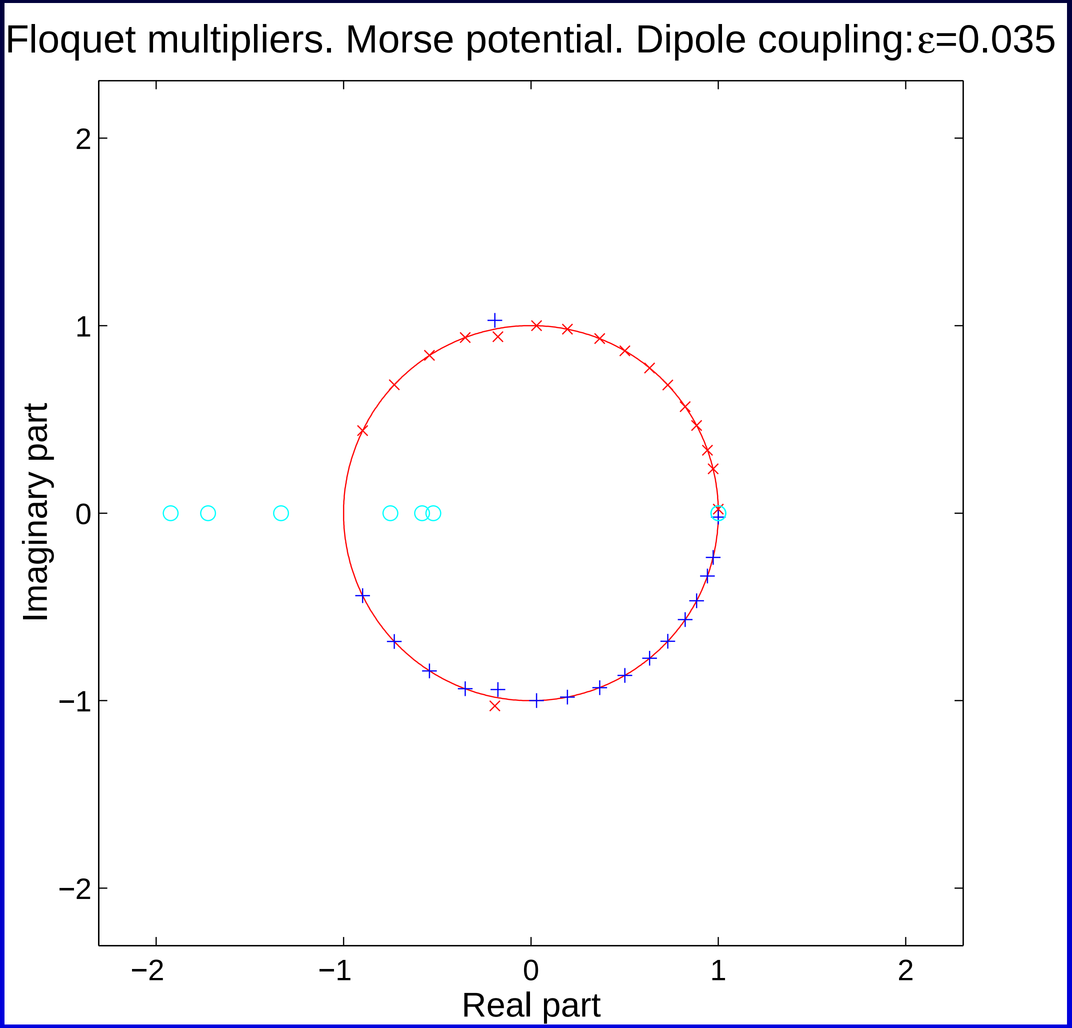
More bifurcations

Krein crunches and subharmonic bifurcations



Corresponding multipliers

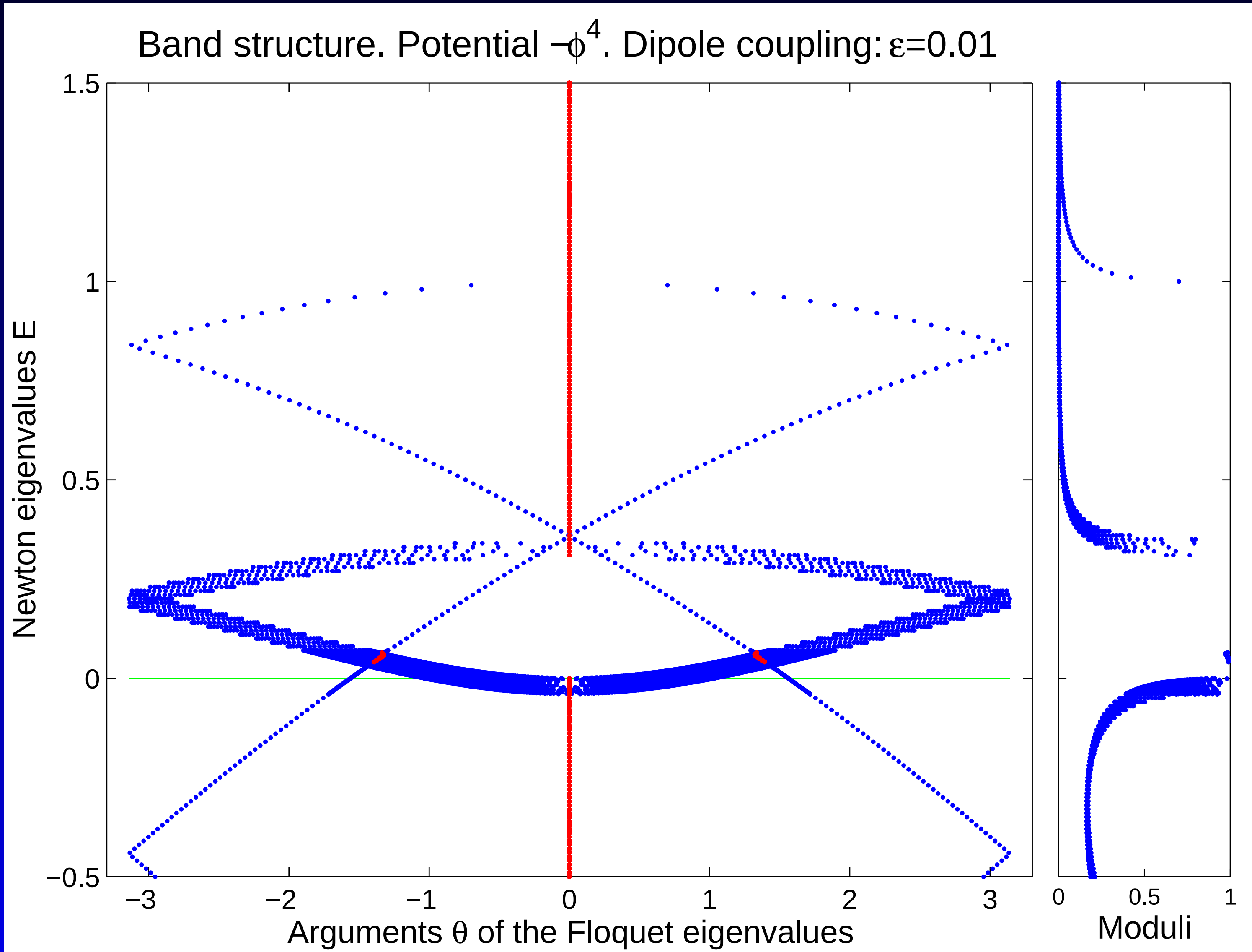
Krein crunches and subharmonic bifurcations



Soft breathers summary

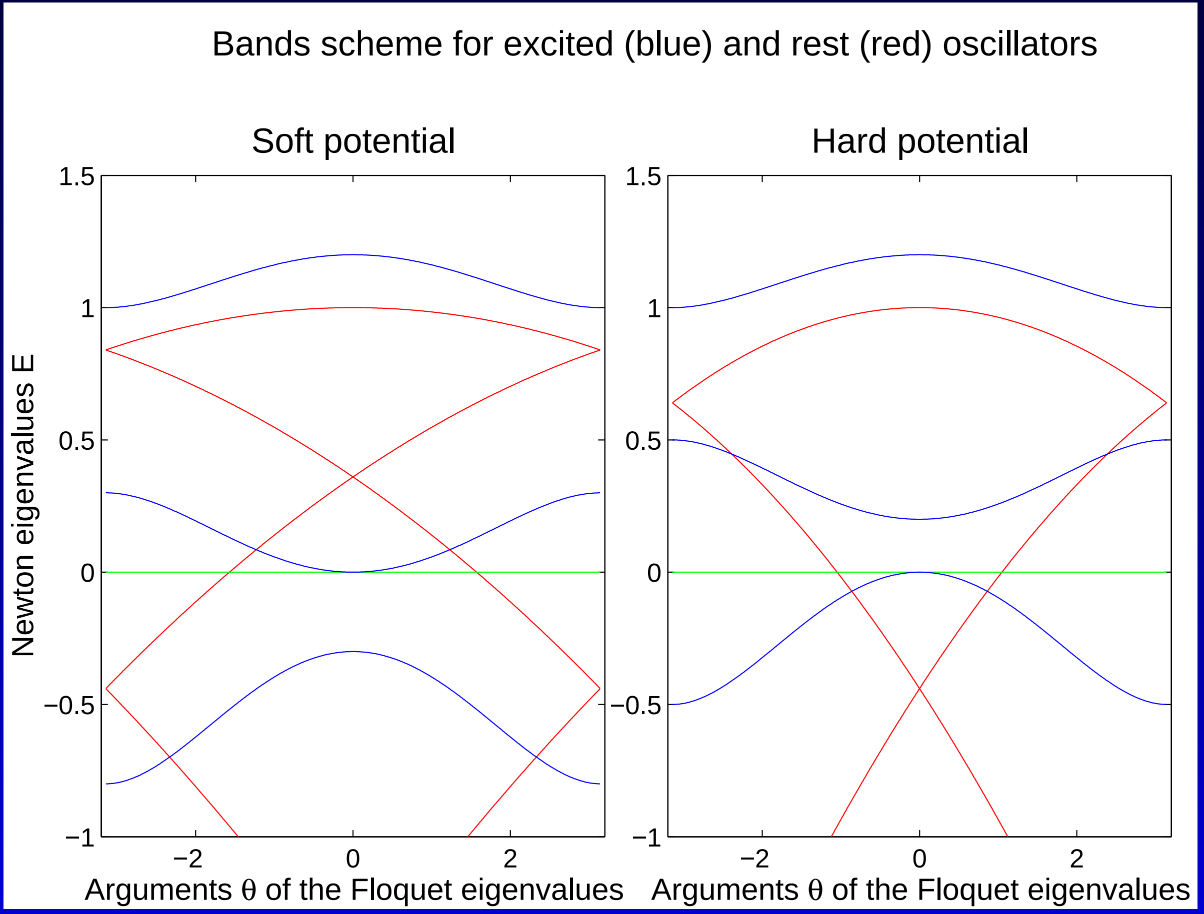
- There are no stable dark breathers with stacking coupling
- There are stable dark breathers with dipole–dipole coupling
- Instability bifurcations
  - Krein crunches (oscillatory instabilities)
    - due to the mixing of the bands
    - smaller in larger systems
  - Subharmonic bifurcations at $\lambda = -1$
    - bring about breather instability
  - With symmetric soft, on–site potential: $V(u_n) = \frac{1}{2}u_n^2 - \frac{1}{4}u_n^4$
    - there are no subharmonic bifurcations

Symmetric soft potential



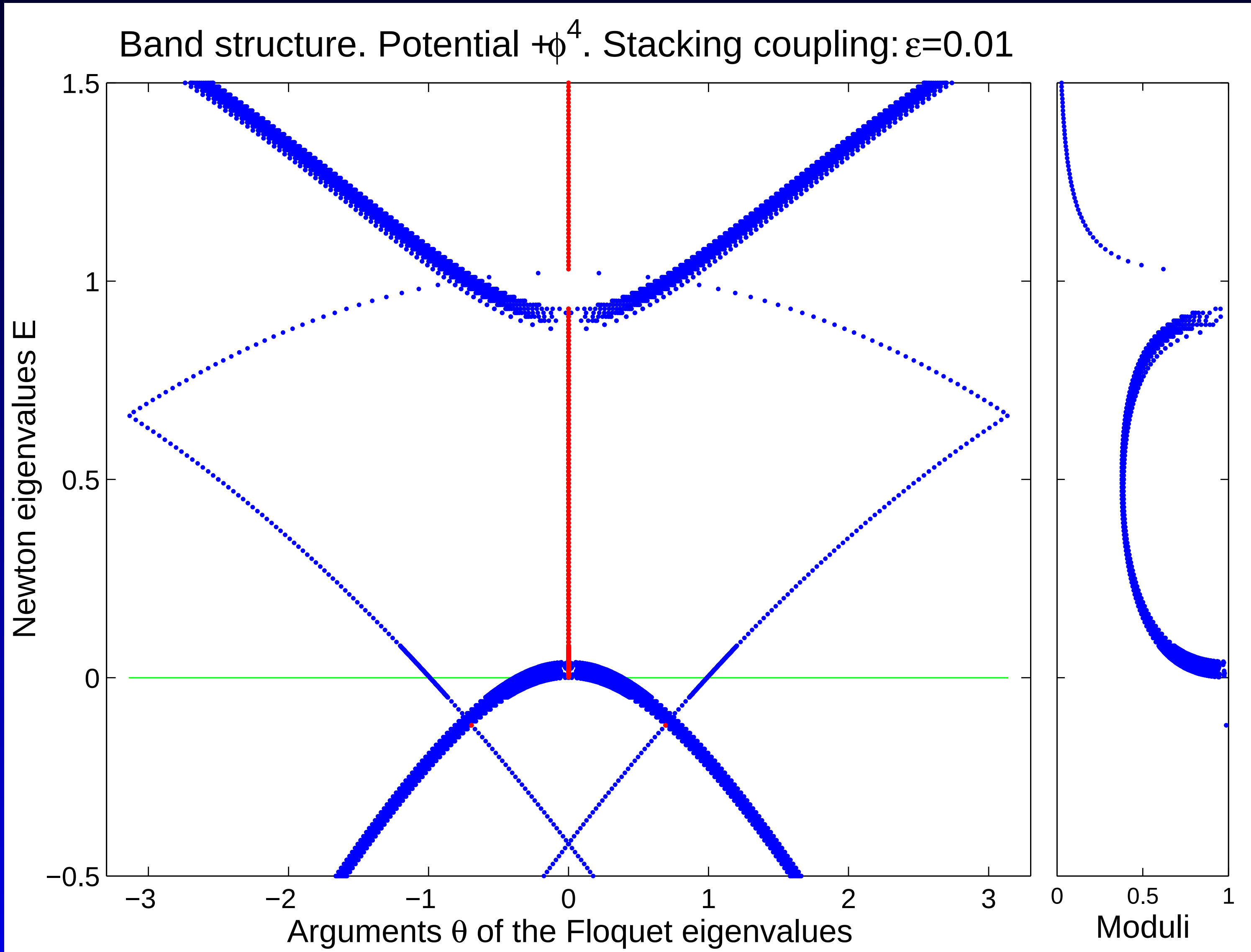
With hard potential

$$\phi^4 : V(u_n) = \frac{1}{2}u_n^2 + \frac{1}{4}u_n^4$$

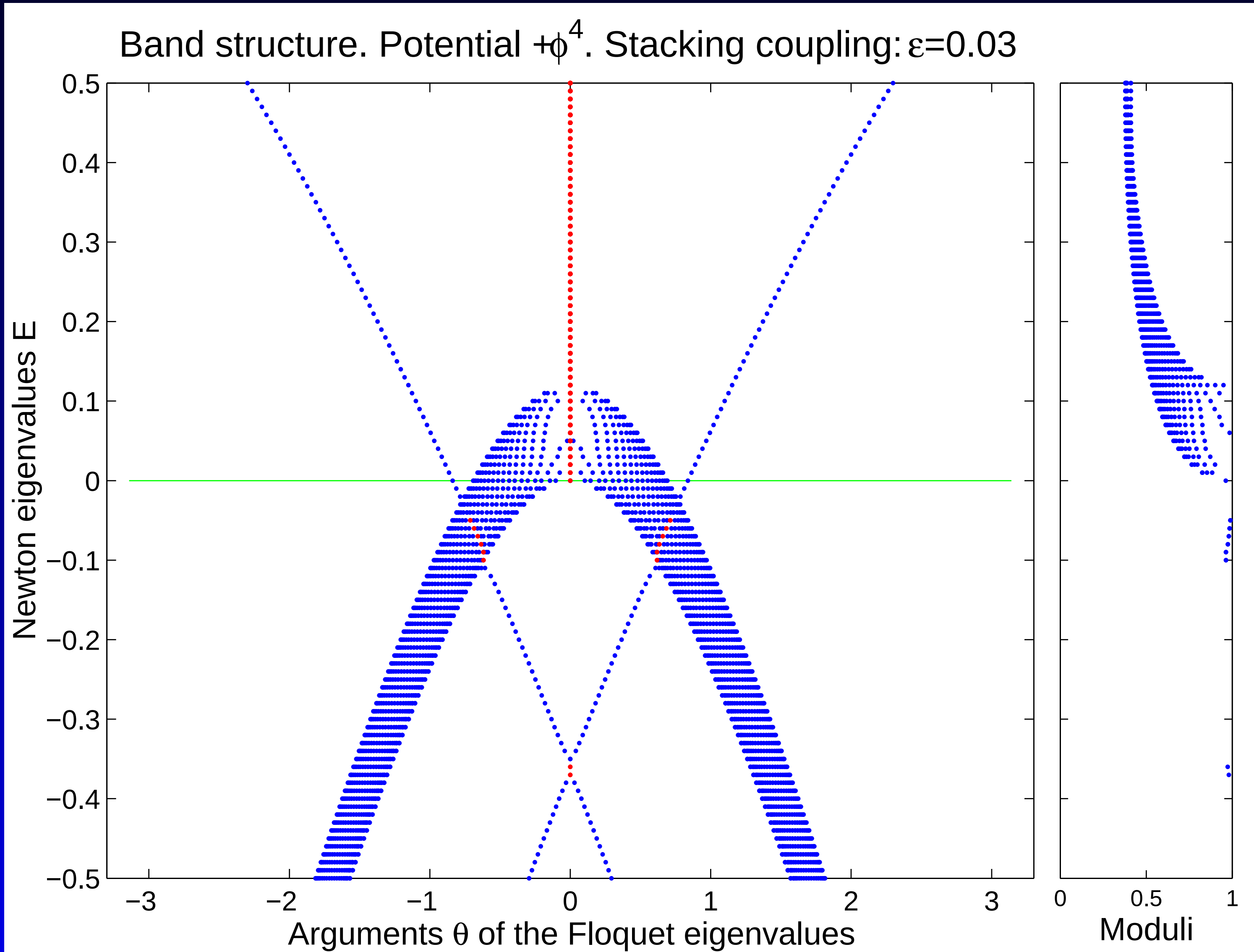


Zero coupling

Hard breather bands



Harmonic instabilities



Hard breathers summary

- There are no stable dark breathers with dipole–dipole coupling
- There are stable dark breathers with stacking coupling
- Instability bifurcations:
  -Harmonic bifurcations at $\lambda = +1$

Future developments

- Evolution of the unstable dark breathers
- Other dark breathers types
- Physical implications

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