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TRAVELING-WAVE SOLUTIONS OF THE SCHWARZ–KORTEWEG–DE VRIES EQUATION IN 2+1 DIMENSIONS AND THE ABLOWITZ–KAUP–NEWELL–SEGUR EQUATION THROUGH SYMMETRY REDUCTIONS

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One of the more interesting solutions of the (2+1)*-dimensional integrable Schwarz–Korteweg–de Vries* (*SKdV*) *equation is the soliton solutions. We previously derived a complete group classification for the SKdV equation in* 2+1 *dimensions. Using classical Lie symmetries, we now consider traveling-wave reductions with a variable velocity depending on the form of an arbitrary function. The corresponding solutions of the* (2+1)*-dimensional equation involve up to three arbitrary smooth functions. Consequently, the solutions exhibit a rich variety of qualitative behaviors. In particular, we show the interaction of a Wadati soliton with a line soliton. Moreover, via a Miura transformation, the SKdV is closely related to the Ablowitz–Kaup–Newell–Segur* (*AKNS*) *equation in* 2+1 *dimensions. Using classical Lie symmetries, we consider traveling-wave reductions for the AKNS equation in* 2+1 *dimensions. It is interesting that neither of the* (2+1)*-dimensional integrable systems considered admit Virasoro-type subalgebras.*

Keywords: partial differential equations, Lie symmetries

1. Introduction

The Schwarz–Korteweg–de Vries (SKdV) equation

$$\frac{\Phi_t}{\Phi_x} + \{\widehat{\Phi}; x\} = 0,$$

where

$$\{\widehat{\Phi}; x\} = \left(\frac{\Phi_{xx}}{\Phi_x}\right)_x - \frac{1}{2}\left(\frac{\Phi_{xx}}{\Phi_x}\right)^2$$

is the Schwarzian derivative, is well known and of great interest in both physics and mathematics [1]. This equation was introduced by Krichever and Novikov in [2] and by Weiss in [3] and is a specialization of the KdV equation that is invariant under the Möbius transformations, i.e., under the group $PSL(2)$.

It is also known that the similarity solutions of integrable nonlinear partial differential equations (PDEs) yield Painlevé transcendents [4], [5]. This connection between the Painlevé equations and soliton-type equations led to the Ablowitz–Ramani–Segur conjecture [6]. Namely, it was demonstrated that ODEs obtained as reductions of the well-known soliton equations yield ODEs with the Painlevé property. A modern survey of this property can be found in [7]. Moreover, similarity reductions of the best-known soliton equations lead to second-order Painlevé equations [4], [8].

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It is interesting to consider the similarity reductions of the Schwartzian equations to ODEs. In [9], the similarity reduction of the SKdV equation was obtained using the scaling group.

Toda and Yu [10] constructed some new integrable models using the Calogero method. In this method, the new equation is derived by considering a Lax pair (L,T) of the basic equation and modifying the T operator to include another spatial dimension z. From the SKdV equation, they thus obtained the equation

$$W_t + \frac{1}{4}W_{xxz} - \frac{W_x W_{xz}}{2W} - \frac{W_{xx}W_z}{4W} + \frac{W_x^2 W_z}{2W^2} - \frac{W_x}{8}\partial_x^{-1}\left(\frac{W_x^2}{W^2}\right)_z = 0, \tag{1}$$

where $\partial_x^{-1} f = \int f\,dx$. Equation (1) is invariant under the Möbius transformation and reduces to the SKdV equation for solutions of the form $W(x,z,t) = \overline{W}(x+z,t)$. In [10], the corresponding Lax pair was presented, and it was proved that it passes the Painlevé test in the sense of the Weiss–Tabor–Carnevale method [11].

The invariance properties of the Kadomtsev–Petviashvili equation [12], the Davey–Stewartson equation [13], and some other physically important integrable nonlinear equations in (2+1)-dimensions have been studied through Lie symmetry analysis. It was shown that all these equations admit infinite-dimensional Lie point groups with a specific Kac–Moody–Virasoro structure. In [14], Senthil Velan and Lakshmanan presented two examples of (2+1)-dimensional integrable equations that do not admit Virasoro-type subalgebras. But they claim that both systems admit a specific type of symmetry. For example, both equations allow infinitesimals up to the quadratic power explicitly in t. They also do not admit any arbitrary function in the infinitesimal variations in t, which in turn leads to the absence of Kac–Moody–Virasoro-type subalgebras.

In this work, we consider the (2+1)-dimensional integrable generalization of SKdV equation (1). Although this (2+1)-dimensional SKdV equation appears in a nonlocal form, using the transformations

$$W = \phi_x, \qquad \phi = e^{\psi}, \qquad \psi_x = u, \qquad \psi_t = v, \tag{2}$$

we can write Eq. (1) as the system of differential equations

$$\begin{aligned} &4u^2 v_x - 4uu_x v + u^2 u_{xxz} - uu_{xx}u_z - 3uu_x u_{xz} + 3u_x^2 u_z - u^4 u_z = 0, \\ &u_t - v_x = 0. \end{aligned} \tag{3}$$

It is interesting that system (3), as well as Eq. (1), was first introduced by Kudriashov and Pickering [15]. Via the Miura transform

$$h_x = \frac{u_{xx}}{4u} - \frac{3u_x^2}{8u^2} - \frac{u^2}{8}, \qquad h_z = -\frac{v}{u},$$

the SKdV is closely related to the equation for h

$$4h_{xt} + h_{xxxz} + 8h_{xz}h_x + 4h_z h_{xx} = 0. \tag{4}$$

This equation is the well-known Ablowitz–Kaup–Newell–Segur (AKNS) equation in 2+1 dimensions. We note that one of the systems considered in [14], the breaking soliton equation, can also be written as the AKNS equation by making a transformation.

In this paper, we derive traveling-wave reductions with a variable velocity depending on the form of an arbitrary function. For this, we apply the classical Lie method to system (3) and also to the related Eq. (4), and we consider the reductions derived from the translation groups and from the infinite-dimensional group.

An interesting feature of our study is that both integrable systems in 2+1 dimensions, the SKdV and the AKNS, admit infinite-dimensional Lie point symmetry groups, but they do not admit Virasoro-type subalgebras.

The invariance study of these reduced systems and (1+1)-dimensional equations and further reductions lead to systems of ODEs and to second-order integrable ODEs. The solutions of all these ODEs are expressible in terms of known functions; some of them can be expressed in terms of the second and third Painlevé transcendents. We also derive exact solutions for the (2+1)-dimensional integrable generalization of the SKdV equation. The appearance of arbitrary functions allows a wide variety of qualitative and physical behaviors for these solutions. In particular, we observe the interaction of a Wadati soliton with different curve solitons.

2. Classical Lie symmetries

To apply the classical method to (2+1)-dimensional system (3), we consider the one-parameter Lie group of infinitesimal transformations in (x,t,z,u,v). The associated Lie algebra of infinitesimal symmetries is the corresponding set of vector fields of the form

$$\mathbf{v} = X\frac{\partial}{\partial x} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v}.$$

We then require that this transformation leave the set of solutions of system (3) invariant. This yields an overdetermined linear system of equations for the infinitesimals $X(x,z,t,u,v)$, $Z(x,z,t,u,v)$, $T(x,z,t,u,v)$, $U(x,z,t,u,v)$, and $V(x,z,t,u,v)$. After the infinitesimals are determined, the symmetry variables are found by solving the invariant-surface conditions

$$\Phi_1 \equiv X\frac{\partial u}{\partial x} + Z\frac{\partial u}{\partial z} + T\frac{\partial u}{\partial t} - U = 0,$$

$$\Phi_2 \equiv X\frac{\partial v}{\partial x} + Z\frac{\partial u}{\partial z} + T\frac{\partial v}{\partial t} - V = 0.$$

Applying the classical method to system (3) yields a system of equations that leads to a four-parameter Lie group. Associated with this Lie group, we have a Lie algebra that can be represented by the generators

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial z},$$

$$\mathbf{v}_3 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}, \qquad \mathbf{v}_4 = t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} - v\frac{\partial}{\partial v}$$

and the infinite-dimensional vector fields of the form

$$\mathbf{v}_\alpha = \alpha(t)\frac{\partial}{\partial x} - \alpha'(t)u\frac{\partial}{\partial v}.$$

The associated algebra $\{\mathbf{v}_\alpha\}$ with $\alpha(t) \in C^\infty(\mathbb{R})$ is not a Virasoro algebra, because the commutation relation between $\mathbf{v}_{\alpha_1}$ and $\mathbf{v}_{\alpha_2}$ is

$$[\mathbf{v}_{\alpha_1}, \mathbf{v}_{\alpha_2}] = 0.$$

To find nonequivalent branches of solutions, we construct the one-dimensional optimal system of subalgebras. The corresponding generators of the optimal system of subalgebras are

$$\langle \mu\mathbf{v}_3 + \mathbf{v}_4 \rangle, \qquad \left\langle \mu\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 + \mathbf{v}_4 \right\rangle, \qquad \langle \mu\mathbf{v}_1 + \mathbf{v}_3 \rangle,$$

$$\langle \mu\mathbf{v}_1 + \mathbf{v}_2 \rangle, \qquad \langle \mu\mathbf{v}_3 \rangle, \qquad \langle \mathbf{v}_4 \rangle, \qquad \langle \mathbf{v}_1 \rangle,$$

where $\mu \in \mathbb{R} \setminus \{0\}$ is arbitrary. In previous papers, we listed the similarity variables and similarity solutions as well as the systems of PDEs obtained when system (3) is reduced using the generators $\{\mathbf{u}_i\}$, $i = 1, \ldots, 7$, which are obtained by adding the infinite-dimensional generator $\mathbf{v}_\alpha$ to the generators of an optimal system.

Our aim in this paper is to use the theory of symmetry reductions to find traveling-wave solutions for the (2+1)-dimensional SKdV equation. For this, we consider the following reductions arising from translations and the infinite-dimensional vector field, i.e., from $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_\alpha$.

Reduction 1. Using the generator $\mu\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$\begin{aligned} &z_1 = x - \frac{1}{\mu}\int \alpha(t)\, dt, \qquad z_2 = \mu z - t, \\ &u = f(z_1, z_2), \qquad v = g(z_1, z_2) - \frac{1}{\mu}\alpha f(z_1, z_2) \end{aligned} \tag{5}$$

and the system of PDEs $\mathbf{S}_1$

$$\begin{aligned} &\mu(-f f_{z_1 z_1} f_{z_2} + 3f_{z_1}^2 f_{z_2} - f^4 f_{z_2} + f^2 f_{z_1 z_1 z_2} - 3f f_{z_1} f_{z_1 z_2}) + 4f^2 g_{z_1} - 4f f_{z_1} g = 0, \\ &g_{z_1} + f_{z_2} = 0. \end{aligned}$$

Reduction 2. Using the generator $\mathbf{v}_1 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$\begin{aligned} &z_1 = x - \int \alpha(t)\, dt, \qquad z_2 = z, \\ &u = f(z_1, z_2), \qquad v = g(z_1, z_2) - \alpha f(z_1, z_2) \end{aligned} \tag{6}$$

and the system of PDEs $\mathbf{S}_2$

$$\begin{aligned} &- f f_{z_1 z_1} f_{z_2} + 3f_{z_1}^2 f_{z_2} - f^4 f_{z_2} + f^2 f_{z_1 z_1 z_2} - 3f f_{z_1} f_{z_1 z_2} - 4f f_{z_1} g = 0, \\ &g_{z_2} = 0. \end{aligned}$$

Reduction 3. Using the generator $\mathbf{v}_2 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$\begin{aligned} &z_1 = z - \frac{x}{\alpha(t)}, \qquad z_2 = t, \\ &u = f(z_1, z_2), \qquad v = g(z_1, z_2) - z\alpha'(t) f(z_1, z_2) \end{aligned}$$

and the system of PDEs $\mathbf{S}_3$

$$\begin{aligned} &- 4\alpha f^2 g_{z_1} + 4\alpha f f_{z_1} g + f^2 f_{z_1 z_1 z_1} - 4f f_{z_1} f_{z_1 z_1} + 3f_{z_1}^3 - \alpha^2 f^4 f_{z_1} = 0, \\ &- \alpha_{z_2} z_1 f_{z_1} + g_{z_1} + \alpha f_{z_2} = 0. \end{aligned}$$

3. Reductions to ODEs and exact solutions

In several cases, the reduced systems of (1+1)-dimensional PDEs admit symmetries that lead to further reductions to systems of ODEs. We again use the techniques of Lie group theory. Moreover, all these systems of ODEs can be reduced to second-order ODEs whose solutions are expressible in terms of known functions, such as the second and third Painlevé transcendents [16].

Two of these systems of (1+1)-dimensional PDEs admit symmetries with arbitrary functions. The corresponding solutions of the (2+1)-dimensional equation involve up to three arbitrary smooth functions.

System $\mathbf{S}_1$ admits the symmetries

$$\mathbf{v}_{11} = \frac{\partial}{\partial z_1}, \qquad \mathbf{v}_\beta = \beta(z_2)\frac{\partial}{\partial z_2} - \beta'(z_2) g \frac{\partial}{\partial g}.$$

Using $c\mathbf{v}_{11} + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = z_1 - c\int \frac{dz_2}{\beta(z_2)}, \qquad f = h, \qquad g = \frac{1}{\beta(z_2)} k(w) \tag{7}$$

and the system of ODEs

$$\begin{aligned} &k' - ch' = 0, \\ &\mu c(4hh'h'' - h^2 h''' - 3(h')^3 + h^4 h') + 4h^2 k' - 4hh'k = 0. \end{aligned}$$

This system is equivalent to the second-order ODE

$$2dh + c\mu(h^4 + (h')^2 - hh'') - k_1 h^3 = 0$$

and

$$k = ch + k_1.$$

The general solution can be written in terms of the Jacobi elliptic functions; consequently, there exist solutions such as

$$h = \frac{a_1}{a_2 + \mathrm{sn}^2(a_4 + a_3 x \mid a_5)},$$

where a_1, a_2, a_3, and a_4 are arbitrary constants and

$$a_5 = -\frac{1}{a_2} - \frac{a_1^2}{4a_2^2 a_3^2 (1 + a_2)}.$$

We have found several particular solutions with a suitable choice of the parameters, and the more interesting ones are

$$\begin{aligned} &h = d_1, &&h = \pm\frac{d_2}{1 + e^{d_2(w - d_1)}}, \\ &h = \pm\frac{2d_2}{(w - d_1)^2 - {d_2}^2}, &&h = \frac{d_2}{\cos\left(d_2(w - d_1)\right)}, \end{aligned} \tag{8}$$

where d_1 and d_2 are arbitrary constants.

Considering transformations (2) as well as the corresponding symmetry reductions (5) and (7), we find from (8) (after some simplifications) that the corresponding family of solutions for (2+1)-dimensional SKdV equation (1) can be written as

$$W = d_1 e^{d_1 x} \rho(z, t), \tag{9}$$

$$W = \frac{\rho(z)}{\cosh^2(d_1 + d_2 x + \varphi(t) + \delta(-t + \mu z))}, \tag{10}$$

$$W = \frac{\rho(z)}{1 + \sin(d_2(d_1 - x + \varphi(t) + \delta(-t + \mu z)))}, \tag{11}$$

where

$$\varphi(t) = \frac{1}{\mu}\int e^{-t/\mu}\alpha(t)\,dt, \qquad \delta(-t+\mu z) = c\int \frac{1}{\beta(z_2)}\,dz_2$$

and $\rho = \rho(z,t)$ are three arbitrary functions.

System $\mathbf{S}_2$ admits the symmetries

$$\mathbf{v}_{21} = \frac{\partial}{\partial z_1}, \qquad \mathbf{v}_{22} = z_1\frac{\partial}{\partial z_1} - f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g},$$
$$\mathbf{v}_\beta = \beta(z_2)\frac{\partial}{\partial z_2} - \beta'(z_2)g\frac{\partial}{\partial g}.$$

Using $c\mathbf{v}_{21} + \mathbf{v}_{22} + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = \log(z_1+c) - \int \frac{dz_2}{\beta(z_2)}, \qquad f = \frac{h(w)}{z_1+c},$$
$$g = \frac{1}{\beta(z_2)}\exp\left(-2\int \frac{dz_2}{\beta(z_2)}\right)k(w)$$

and the system of ODEs

$$k' = 0,$$
$$(4hh' - 4h^2)ke^{2w} - 4hh'h'' + h^2h''' + 3(h')^3 - h^4h' = 0.$$

Changing the variables as $e^{2w} = z$, $h(w) = Y(z)$, dividing by $u^4 z$, and integrating once with respect to z, we obtain

$$Y'' = \frac{(Y')^2}{Y} + \frac{1}{4z^2}Y^3 + \frac{k_1}{8z^2}Y^2 - \frac{Y}{z} + \frac{k}{2z} = 0.$$

This equation is the Painlevé III equation (1906)

$$Y'' = \frac{(Y')^2}{Y} + \frac{\alpha Y^2 + \gamma Y^3}{4z^2} + \frac{\beta}{4z} + \frac{\delta}{4Y}$$

with $\alpha = k_1/2$, $\beta = 2k$, $\gamma = 1$, and $\delta = 0$.

Using $\mathbf{v}_{21} + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = z_1 - \int \frac{dz_2}{\beta(z_2)}, \qquad f = h, \qquad g = \frac{1}{\beta(z_2)}k(w)$$

and the system of ODEs

$$k' = 0,$$
$$-4hh'h'' + h^2h''' + 3(h')^3 - h^4h' + 4h^2k' - 4hh'k = 0.$$

This system can be transformed to the second-order autonomous ODE

$$h'' = \frac{3}{2}\frac{(h')^2}{h} + \frac{h^3}{2} - \frac{k_2}{2}h + 4c$$

and

$$k = c.$$

The solution can be written in terms of the elliptic functions.

Using $\mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = z_1, \qquad f = h, \qquad g = \beta(z_2)c. \tag{12}$$

If we set $c = 0$, then h becomes arbitrary.

Considering transformations (2) as well as the corresponding symmetry reductions (6) and (12), we find that the corresponding family of solutions of (2+1)-dimensional SKdV equation (1) is

$$W = \rho(z)f(x - \varphi(t)),$$

where ρ and f are arbitrary functions.

System $\mathbf{S}_3$ admits the symmetries

$$\mathbf{v}_{31} = \beta(z_2)\frac{\partial}{\partial z_1} + (\alpha\beta' + \alpha'\beta)f\frac{\partial}{\partial g},$$
$$\mathbf{v}_{32} = -\frac{\alpha'(z_2)}{\alpha^2(z_2)}z_1\frac{\partial}{\partial z_1} + \frac{1}{\alpha(z_2)}\frac{\partial}{\partial z_2} + \frac{\alpha'(z_2)}{\alpha^2(z_2)}g\frac{\partial}{\partial g}.$$

Using $\mathbf{v}_{32}$, we obtain the similarity variable and similarity solutions

$$w = z_1\alpha(z_2), \qquad f = h, \qquad g = \frac{1}{z_1}k(w)$$

and the system of ODEs

$$k - wk' = 0,$$
$$w^2h^2h''' - 4w^2hh'h'' + 3w^2(h')^3 - w^2h^4h' - 4wh^2k' + 4whh'k + 4h^2k = 0.$$

This system is equivalent to the second-order autonomous ODE

$$h'' = \frac{3}{2}\frac{(h')^2}{h} + \frac{h^3}{2} - k_2h + 4k_1$$

and

$$k = k_1w.$$

The solution can be written in terms of the elliptic functions. A particular solution for the SKdV equation is

$$W = \frac{\rho(z)}{(2 + c_1(c_2 + x + z\alpha(t)))^2}.$$

4. Classical Lie symmetries of the (2+1)-dimensional AKNS equation

To study the invariance properties, we consider AKNS equation (4). The invariance of Eq. (4) under the infinitesimal point transformations in (x, t, z, h) with the associated set of vector fields of the form

$$\mathbf{v} = X\frac{\partial}{\partial x} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + H\frac{\partial}{\partial h}$$

leads to the following infinite-dimensional Lie algebra of symmetries.

Applying the classical method to Eq. (4) yields a system of equations that leads to a six-parameter Lie group. Associated with this Lie group, we have a Lie algebra that can be represented by the generators

$$\begin{aligned}
&\mathbf{v}_1 = \frac{\partial}{\partial t}, && \mathbf{v}_2 = \frac{\partial}{\partial z},\\
&\mathbf{v}_3 = x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, && \mathbf{v}_4 = tx\frac{\partial}{\partial x} + 2tz\frac{\partial}{\partial t} + 2t^2\frac{\partial}{\partial z} + (xz - tu)\frac{\partial}{\partial u},\\
&\mathbf{v}_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial u}, && \mathbf{v}_6 = -x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}
\end{aligned}$$

and infinite-dimensional vector fields of the form

$$\mathbf{v}_\alpha = \alpha(t)\frac{\partial}{\partial x} + \alpha'(t)z\frac{\partial}{\partial u}, \qquad \mathbf{v}_\beta = \beta(t)\frac{\partial}{\partial u}.$$

It is interesting that the associated algebra does not contain a Virasoro algebra.

Our aim is to use the theory of symmetry reductions to find traveling-wave solutions of the (2+1)-dimensional AKNS equation. For this, we consider the following reduction arising from translations and the infinite-dimensional vector field, i.e., from $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_\alpha$.

Reduction. Using the generator $\mu\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$z_1 = x - \int \alpha(t)\,dt, \qquad z_2 = z - \mu t, \qquad h = \alpha z_2 + f(z_1, z_2)$$

and the PDE

$$4\mu f_{z_1z_2} - 4f_{z_1z_1}f_{z_2} - 8f_{z_1}f_{z_1z_2} - f_{z_1z_1z_1z_2} = 0. \tag{13}$$

5. Reduction to an ODE and exact solutions

The reduced (1+1)-dimensional PDE admits symmetries with arbitrary functions that lead to further reductions to an ODE. The corresponding solutions of the (2+1)-dimensional equation involve up to three arbitrary smooth functions.

Equation (13) admits the symmetries

$$\begin{aligned}
&\mathbf{v}_1 = \frac{\partial}{\partial z_1}, && \mathbf{v}_2 = \frac{\partial}{\partial f},\\
&\mathbf{v}_3 = z_1\frac{\partial}{\partial z_1} + (\mu z_1 - f)\frac{\partial}{\partial f}, && \mathbf{v}_\beta = \beta(z_2)\frac{\partial}{\partial z_2}.
\end{aligned}$$

Using $c\mathbf{v}_1 + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solution

$$w = z_1 - \int \frac{dz_2}{\beta(z_2)}, \qquad f = z_1 + g(w)$$

and the ODE

$$g^{(\mathrm{IV})} + (8 - 4\mu)g'' + 12g'g''.$$

Integrating once with respect to w and using $y = g'$, we obtain the second-order ODE

$$y'' + 6y^2 + (8 - 4\mu)y - c = 0.$$

The solutions can be written in terms of the elliptic functions. We have found several particular solutions with a suitable choice of the parameters; some of them are

$$\begin{aligned}
&g = c_1 w + c_2,\\
&g = \frac{\mu - 2}{3}w + \frac{1}{w + c_1} + c_2,\\
&g = \frac{\mu - 2 - c_1^2}{3}w + c_1 \tanh(c_1 w + c_2) + c_3.
\end{aligned}$$

The corresponding solutions of the AKNS equation are

$$\begin{aligned}
h &= c_2(x - \alpha(t)) + \beta(z - \mu t) + \alpha'(t)(z - \mu t),\\
h &= \frac{\mu - 2}{3}(x - \alpha(t) - \beta(z - \mu t)) + x - \alpha(t) +\\
&\quad + \alpha'(t)(z - \mu t) + \frac{1}{x - \alpha(t) - \beta(z - \mu t) + c_1} + c_2,\\
h &= \frac{\mu - 2 - c_1^2}{3}(x - \alpha(t) - \beta(z - \mu t)) + x - \alpha(t) + \alpha'(t)(z - \mu t) +\\
&\quad + c_1 \tanh(c_1(x - \alpha(t) - \beta(z - \mu t)) + c_2) + c_3.
\end{aligned}$$

6. Some explicit solutions

The appearance of the arbitrary functions in the solutions allows a wide variety of qualitative and physical behaviors for these solutions. With appropiate choices for the arbitrary functions, we previously [16], [17] exhibited large families of solitary waves, coherent structures as n-soliton bound states, kinks, and a great variety of stationary solutions such as multidromions. By choosing f as a compact support C^∞, we also obtained compactons.

The arbitrary function $\rho(z, t)$ in solution (9) can represent any interaction process in 1+1 dimensions; moreover, any sum of coherent structures in 1+1 dimensions provides new solutions. These last solutions describe an elastic collision. In particular, we can choose a great variety of solutions. As an example, we

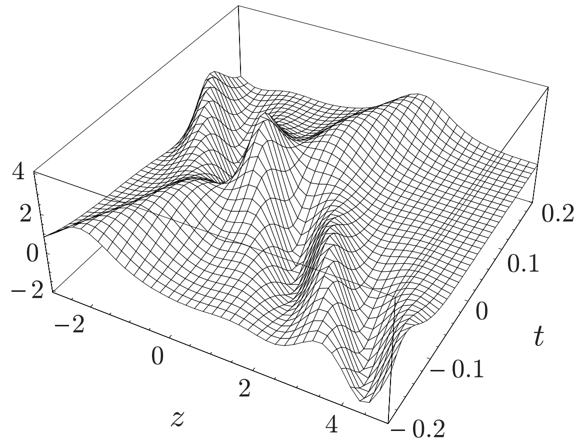

Fig. 1. Wadati soliton and line soliton.

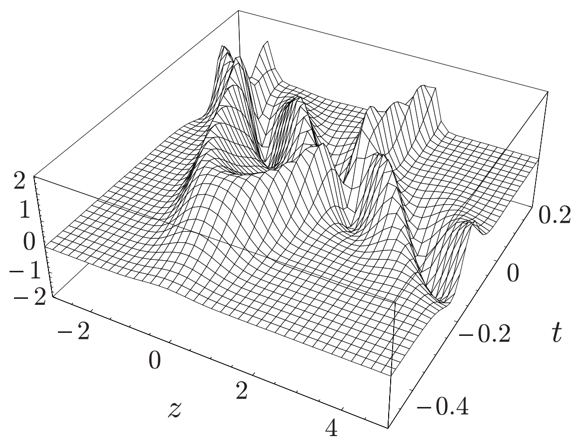

Fig. 2. Wadati soliton and parabolic soliton.

can observe the interaction modulated by e^{ax} of a Wadati soliton [18] in z given by

$$2\partial_z\left(\arctan\left(\frac{c\sin(nz+dt)}{n\cosh(cz+et)}\right)\right)$$

with a line soliton in (z, t).

In Fig. 1, we can observe the interaction of a Wadati soliton in (z, t) for $n = 3$ and $c = 1$ with a line soliton that evolves over $z = 10t$.

In Fig. 2, we can observe the interaction of a Wadati soliton with a parabolic soliton that evolves over the curve $z^2 - 10t - 3 = 0$.

In Fig. 3, for $t = -0.2$, we can see the interaction of a Wadati soliton with a line soliton that evolves over $z = 10t$ modulated by $e^{x/10}$ in the xz plane.

We can similarly consider kinks, multidromions, etc., in (z, t) modulated by e^{ax}.

7. Conclusions

We have discussed symmetry reductions and exact solutions of the (2+1)-dimensional integrable generalization of the SKdV equation and also the AKNS equation in 2+1 dimensions. This last equation is closely related to the SKdV equation via a Miura transformation. Using classical Lie symmetries, we have

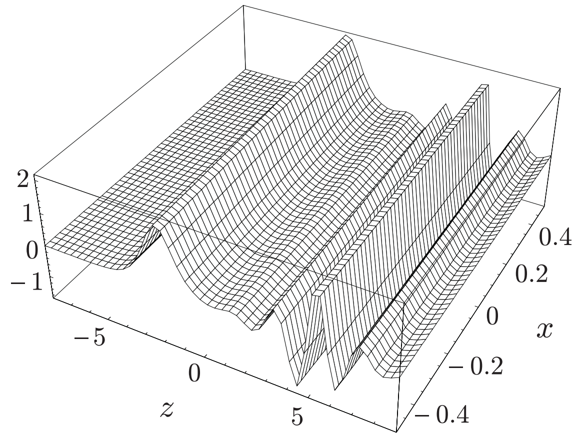

Fig. 3. Wadati soliton and line soliton, $t = -0.2$.

considered traveling-wave reductions for these integrable equations in 2+1 dimensions. It is interesting that the two integrable equations considered in 2+1 dimensions do not admit Virasoro-type subalgebras.

Using the classical Lie method, we have obtained systems of PDEs in 1+1 dimensions and systems of ODEs; by further reductions, we have obtained second-order integrable ODEs whose solutions are all expressible in terms of known functions, some of them expressible in terms of the second and third Painlevé transcendents. For the SKdV in 2+1 dimensions, we have obtained families of solutions that have a rich variety of qualitative behaviors. This results from the freedom in choosing the arbitrary functions as in $e^x\rho(z,t)$.

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