Nonlinear Double Day
Sevilla, Spain, May 17-18, 2004

BREATHERS WITH SMALL AND LARGE AMPLITUDES IN FPU LATTICES

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- Fermi-Pasta-Ulam lattice: a set of particles with a NON-LINEAR interaction potential $\rightarrow$ No anticontinuous limit

  Hamiltonian:

  $$H = \sum_n \left(\frac{1}{2} \dot{x}_n^2 + V(x_n - x_{n-1}) \right)$$

  Dynamical equations:

  $$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}) \, , \quad n \in \mathbb{Z}$$

- Flach (1995): proof of breathers existence for potentials $V(x) = x^{2m}$, $m \geq 2$

- Livi et al. (1997) proof for diatomic fpu chains

- Aubry et al. (2001) rigorous proof when V is a convex polynomial of degree 4

- G. James (2001): another proof based on a centre manifold technique.

  Small amplitude breathers above the phonon band exist/not exist if V satisfies/violates a local hardening condition:

  $$B = \frac{1}{2}V^{(4)}(0) - (V^{(3)}(0))^2 > 0$$

  B can be interpreted as a hardening coefficient since breathers with amplitude $A \approx 0$ have frequency $\omega_b \approx 2 + \frac{B}{8}A^2$

MAIN RESULTS OF THE CENTRE MANIFOLD METHOD

For ω_b slightly above the phonon band all small time periodic solutions verify:

$$y_n = (-1)^n \xi_n \cos(\omega_b t) + \text{h.o.t.}$$

where:

- $y_n = V'(x_n)$ is the force
- ξ_n satisfies the recurrence relation: $\xi_{n+1} + \xi_{n-1} - 2\xi_n = \mu \xi_n - B\xi_n^3 + \text{h.o.t.}$
- and $\mu = \omega_b^2 - 4 << 1$

Breathers are homoclinic solutions to 0 of the recurrence relation satisfying

$$\lim_{n \to \pm\infty} \xi_n = 0$$

With $\xi_n = \sqrt{\frac{\mu}{B}}\ v(n\sqrt{\mu})$ we can approximate the recurrence relation by the differential equation

$$v'' = v - v^3 \quad ,$$

which has the homoclinic solutions
$v(x) = \pm\sqrt{2}/\cosh(x+c)$

Thus, we get the following approximations to the exact solutions:

$$y_n^1(t) \simeq (-1)^n \sqrt{\frac{2\mu}{B}} \frac{\cos\omega_b t}{\cosh(n\sqrt{\mu})}$$

$$y_n^2(t) \simeq (-1)^n \sqrt{\frac{2\mu}{B}} \frac{\cos\omega_b t}{\cosh(\,(|n+1/2|-1/2)\sqrt{\mu}\,)}$$

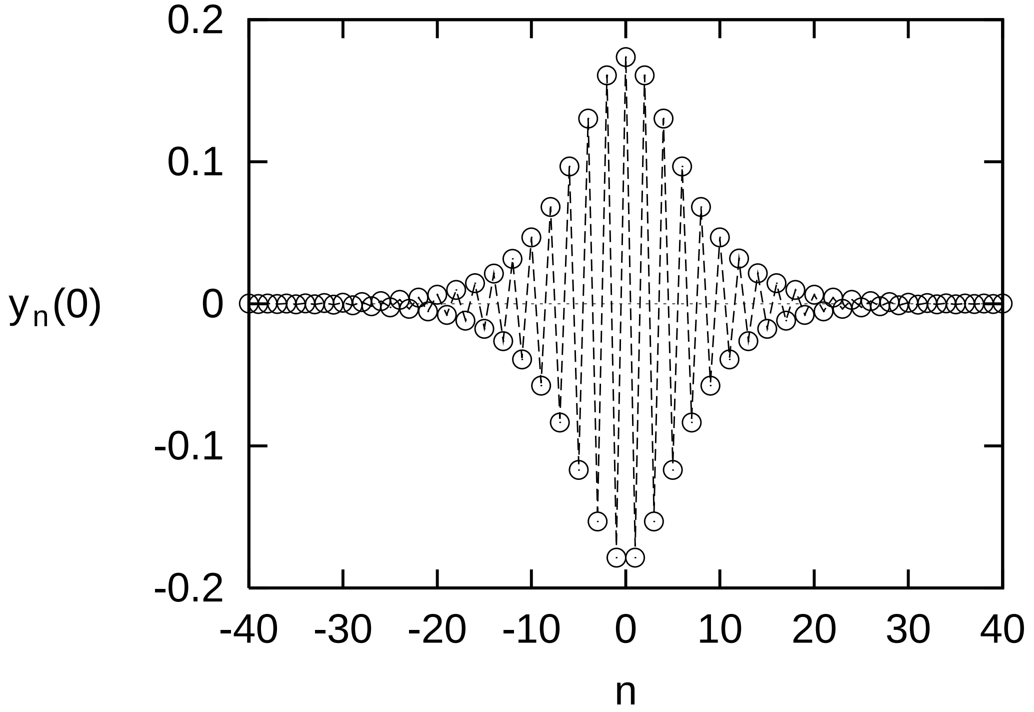
$y_n^1(t) = y_{-n}^1(t)$ is the site-centred mode
$y_n^2(t) = -y_{-n-1}(t)$ is the bond-centred mode

CONSEQUENCES: as $\omega_b \to 2^+$ SAB have

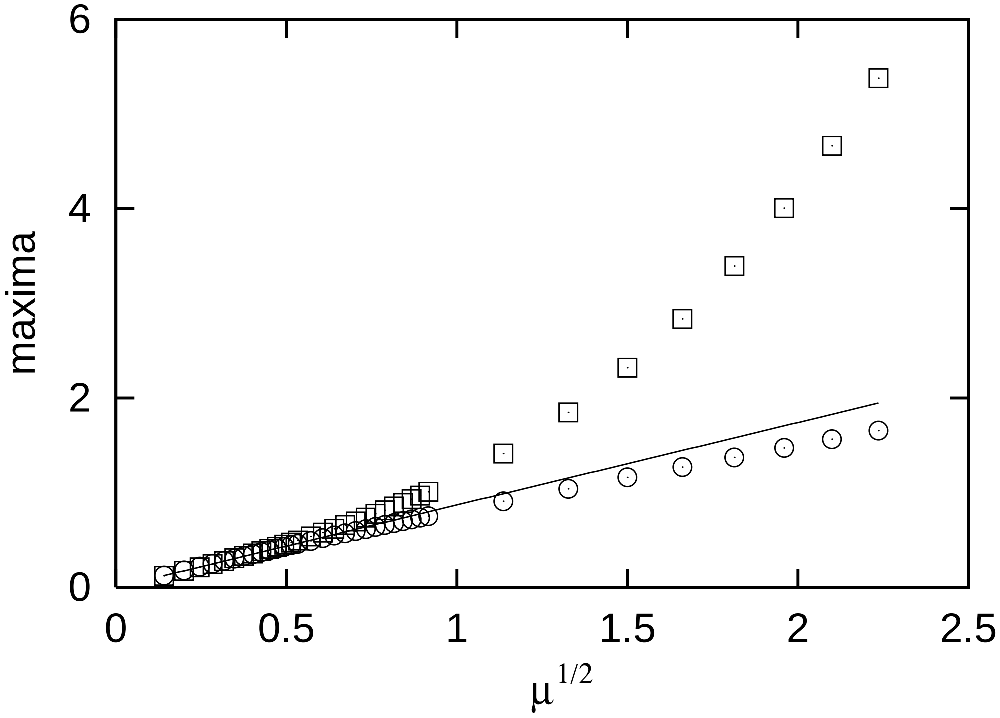
- Maximum amplitude $A \approx \sqrt{\frac{2\mu}{B}}$
- Width is $O(|\omega_b - 2|^{-1/2})$
- If $B > 0$ breathers exist for arbitrary small values of energy

NUMERICAL TEST

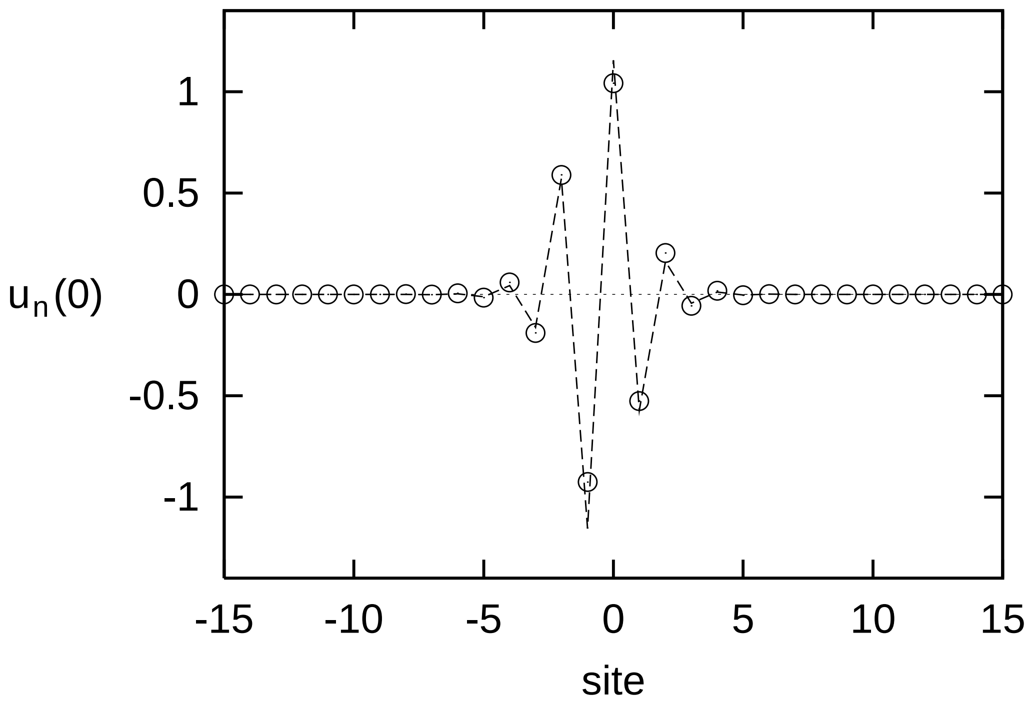
- Consider $V(u) = \frac{u^2}{2} + \frac{K_3}{3}u^3 + \frac{u^4}{4}$ so that $B = 3 - 4K_3^2 > 0$ if $|K_3| < \sqrt{3}/2 \simeq 0.86$
- Difference displacement variables: $u_n = x_n - x_{n-1}$, one-to-one related to the force y_n.
- Periodic boundary conditions



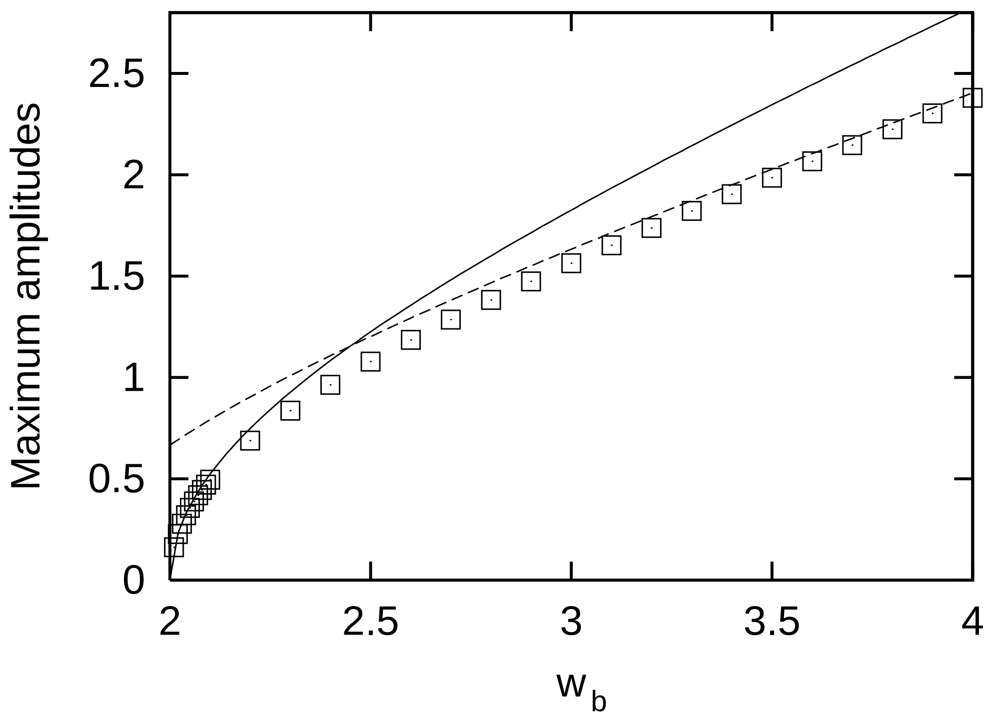
0.2
0.1
0
-0.1
-0.2
$y_n(0)$
-40
-30
-20
-10
0
10
20
30
40
n



6
4
2
0
maxima
0
0.5
1
1.5
2
2.5
$\mu^{1/2}$



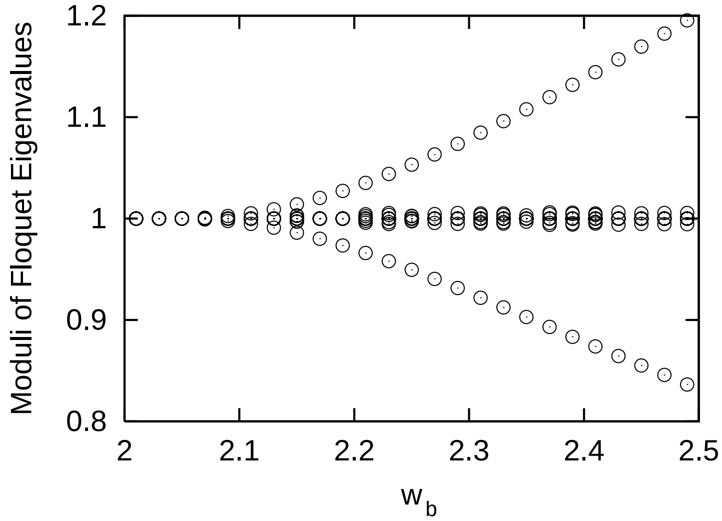
1
0.5
u_n(0)
0
-0.5
-1
-15
-10
-5
0
5
10
15
site



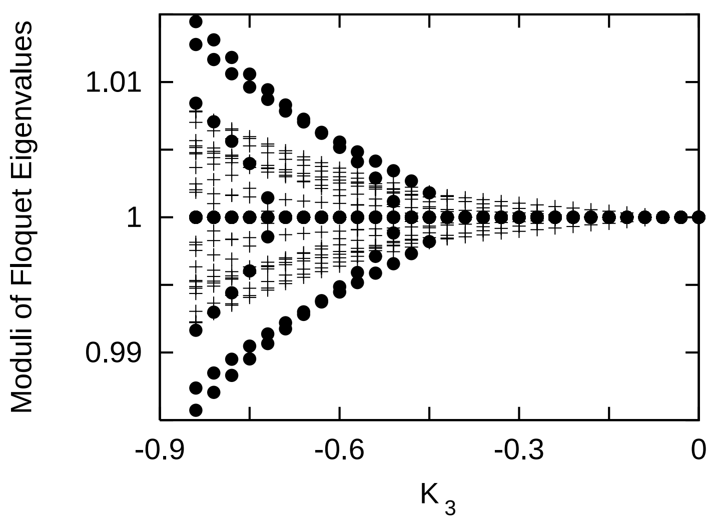
Maximum amplitudes
2.5
2
1.5
1
0.5
0
2
2.5
3
3.5
4
w_b

WHAT ABOUT STABILITY?

- The Sievers-Takeno mode has a harmonic instability

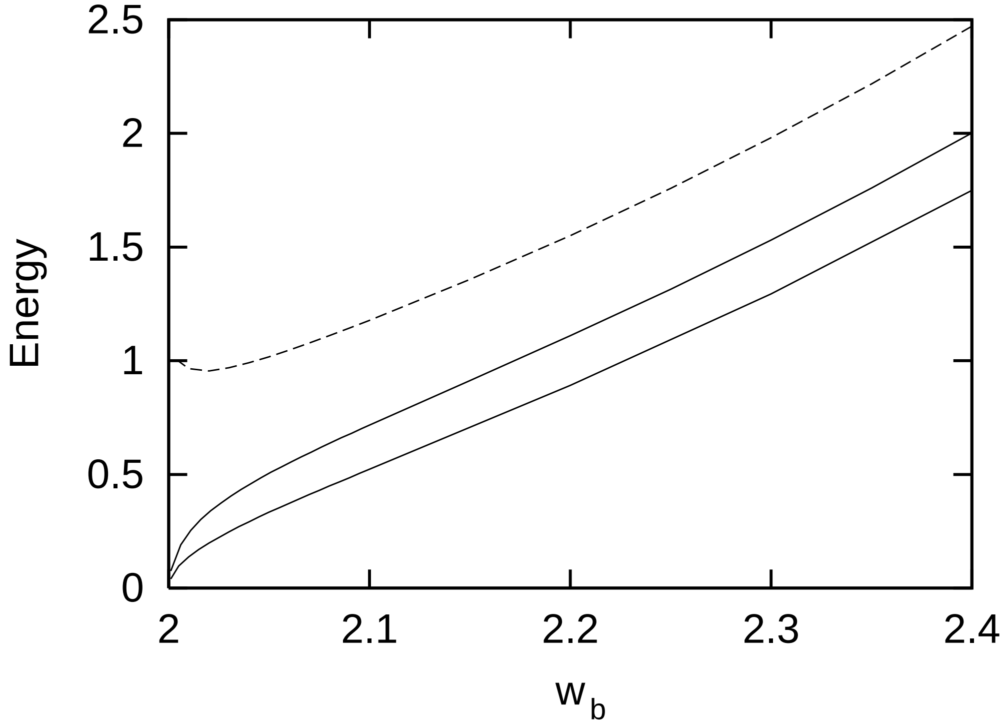


- Non-even potentials induce oscillatory instabilities



WHAT HAPPENS IN THE PARAMETER REGION $B < 0$?

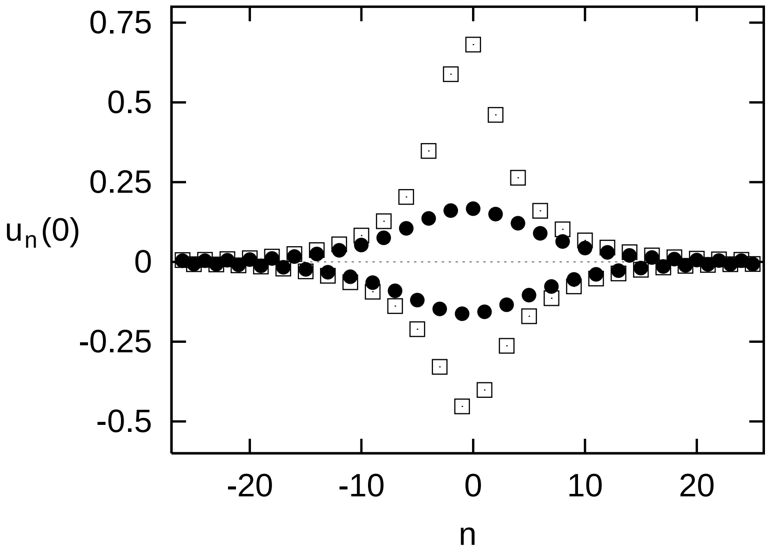
Breathers exist but their amplitudes DO NOT tend to zero as $\omega_b \to 2^+$



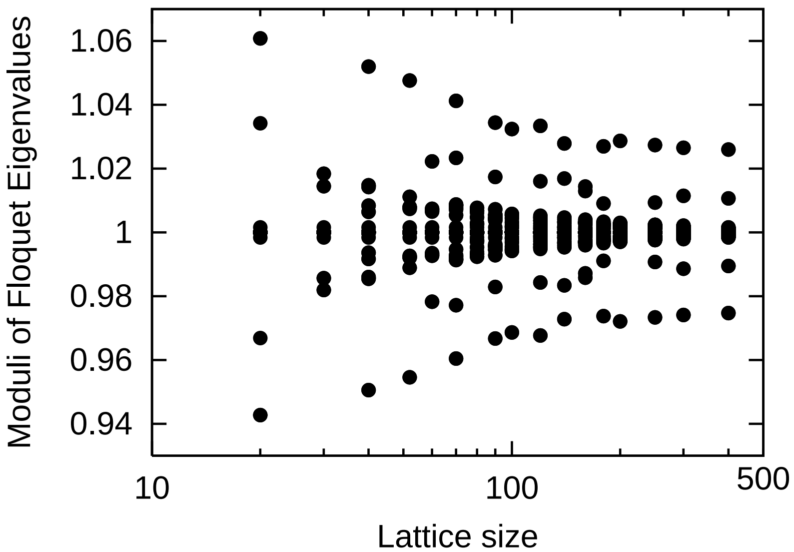
Thus, there is an energy threshold for breather creation in these fpu systems

Other properties:

- LAB have an exponential decay



- Same stability properties as SAB

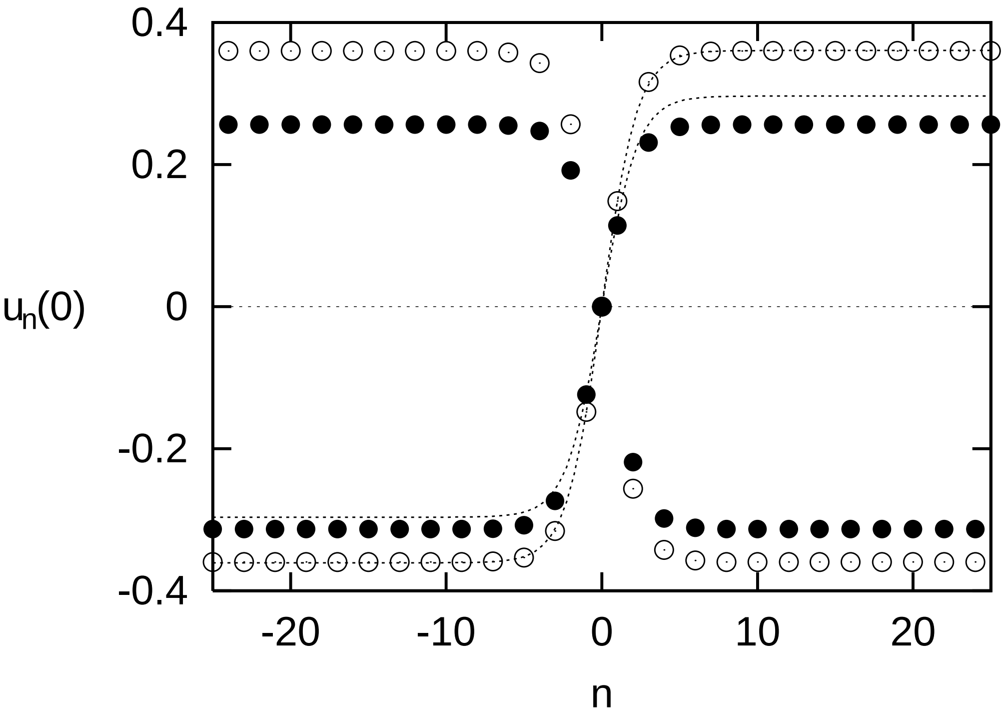


DARK BREATHERS

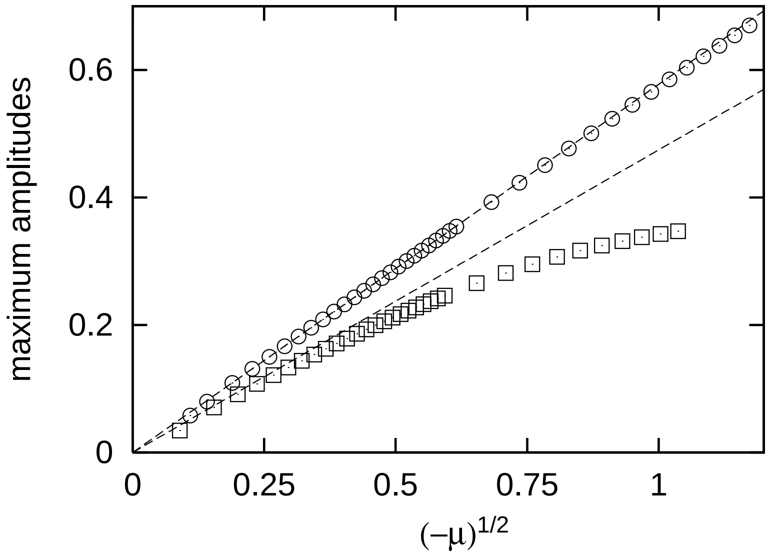
For $B < 0$ and $\mu < 0$ there exist small amplitude heteroclinic solutions of the recurrence relation connecting two nonlinear phonons at infinity.

Approximation to these solutions: the variable change $\xi_n = \sqrt{-\mu}\ \rho_n$ leads to the integrable differential equation $v'' = -v - Bv^3$, so that we get

$$y_n^1(t) \simeq (-1)^n \sqrt{\frac{\mu}{B}} \tanh\left(\frac{n\sqrt{-\mu}}{\sqrt{2}}\right) \cos(\omega_b t)$$

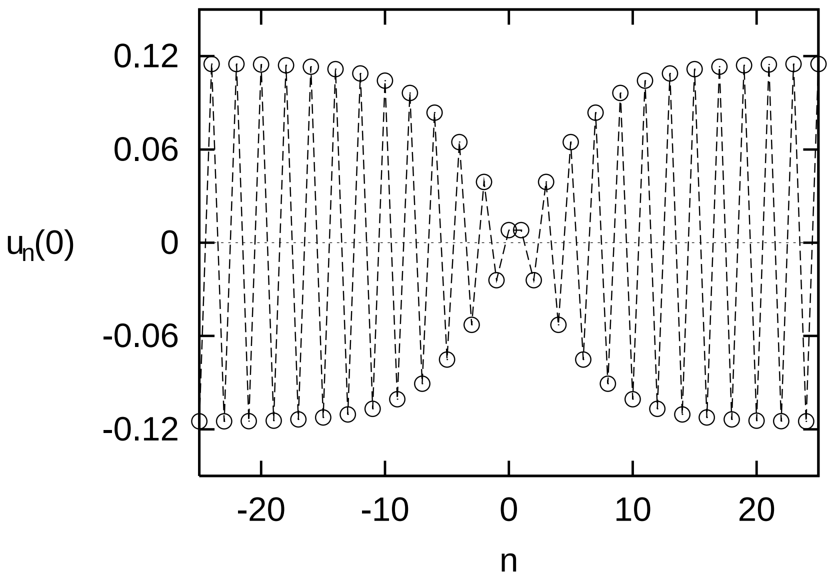


- Continuation decreasing ω_b

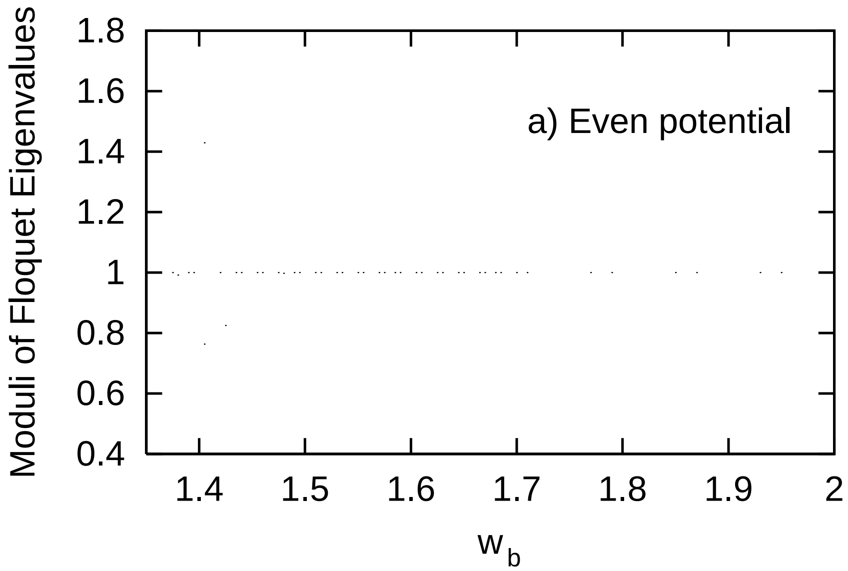


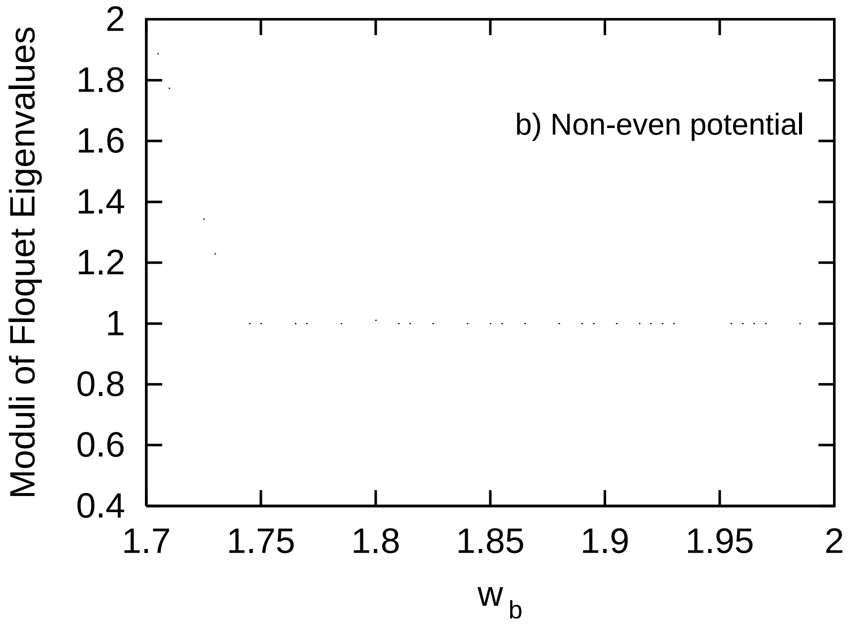
- Bond centred dark mode:

$$y_n(t) \simeq (-1)^n \sqrt{\frac{\mu}{B}} \tanh\left(\frac{(-n+1/2)\sqrt{-\mu}}{\sqrt{2}}\right) \cos(\omega_b t)$$

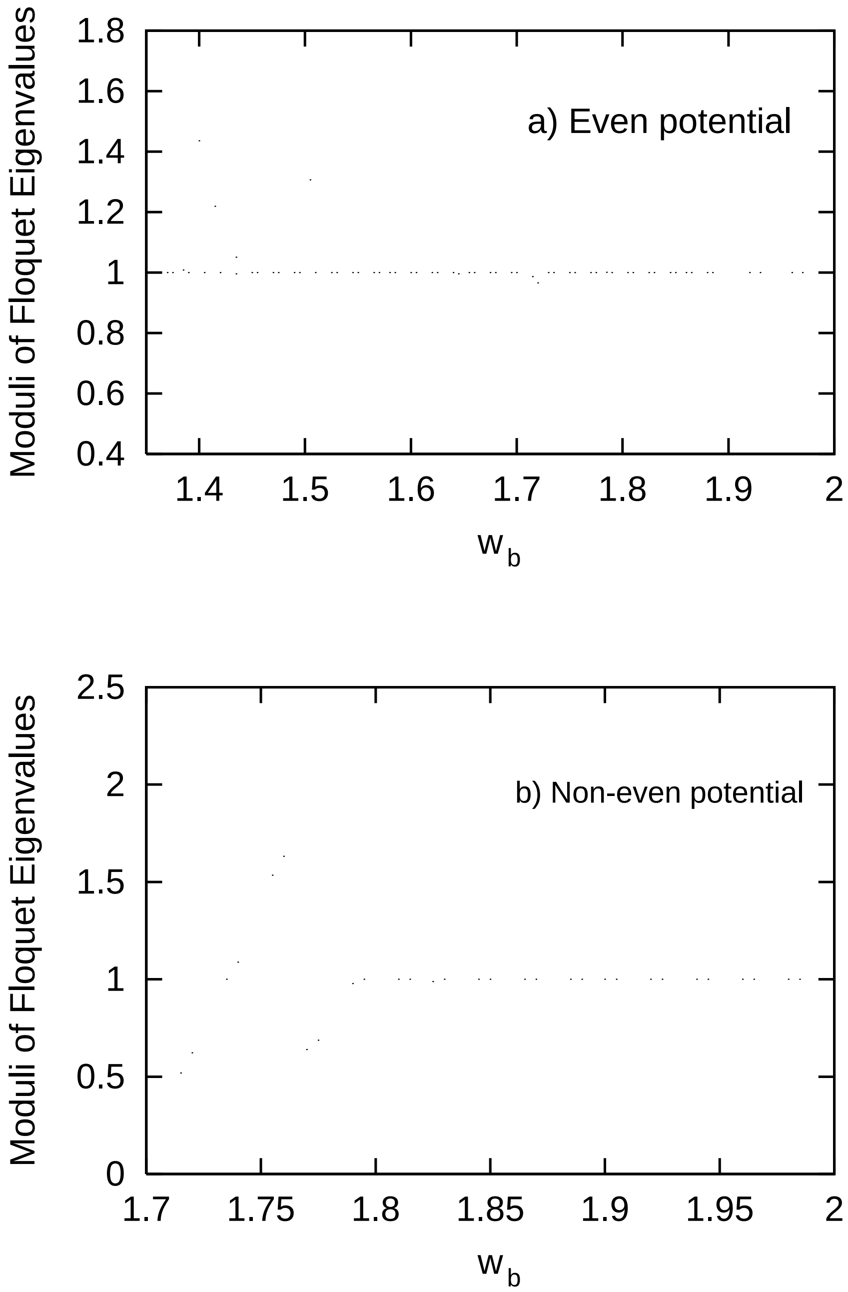


STABILITY OF THE SITE-CENTRED MODE

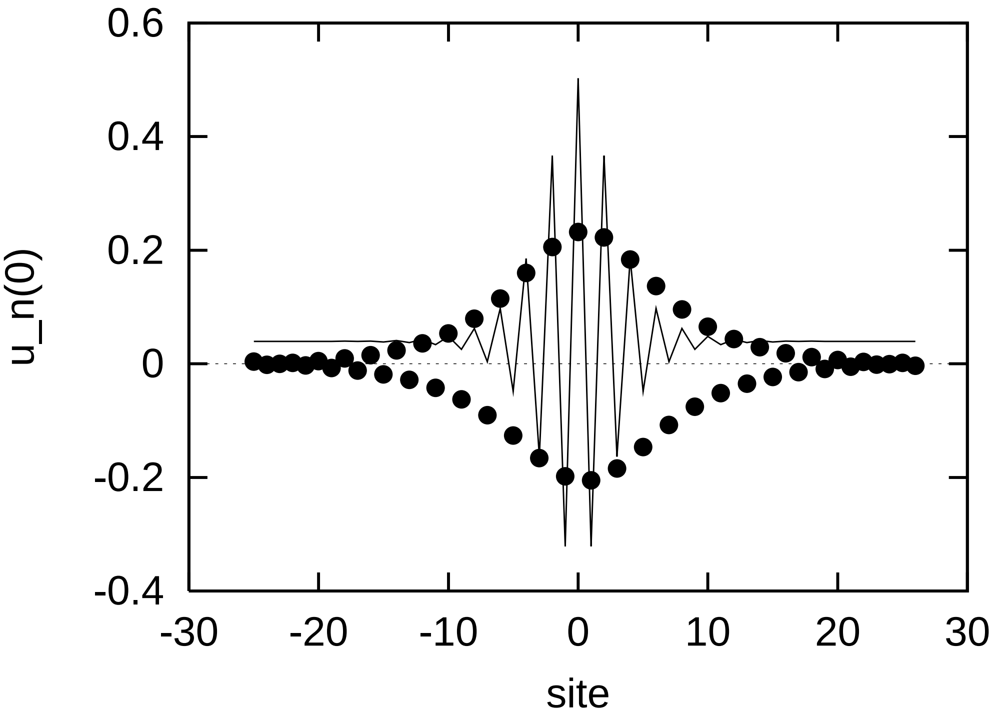




STABILITY OF THE BOND-CENTRED MODE



BREATHERS WITH AN UNIFORM STRESS



Breathers which verify

$$\lim_{n\to\pm\infty} u_n = x_n - x_{n-1} = c$$

can be seen as localized time-periodic oscillations superposed onto a uniformly stressed static state given by $x_n = cn$

Introduce the change of variable

$$x_n(t) = cn + \tilde{x}_n(t\sqrt{V''(c)}) \text{ where } \tilde{x}_n \ll 1$$

and the modified potential

$$\tilde{V}(u) = (V(c+u) - V'(c)\,u)/V''(c)$$

Then the dynamical equations are

$$\frac{d^2}{dt^2}\tilde{x}_n = \tilde{V}'(\tilde{x}_{n+1} - \tilde{x}_n) - \tilde{V}'(\tilde{x}_n - \tilde{x}_{n-1})\,, \quad n \in \mathbb{Z}$$

All analytical and numerical results above presented readily apply to this new family of solutions with modified parameters
$\mu = \omega_b^2 - 4\,V''(c)$,
$B(c) = \frac{1}{2}V''(c)\,V^{(4)}(c) - (V^{(3)}(c))^2$.

CONCLUSIONS

- "Bright" and "Dark" Breathers are well described, even far from the phonon band, by approximate analytical expressions.

- We have determined more precisely the parameter region where a family of LAB with an energy threshold exists. This is a rare and relatively unexplored phenomenon in one-dimensional lattices.

- We have studied the stability of the site-centred and bond-centred modes in both bright and dark cases. In particular, we have checked that non-even potentials induce oscillatory instabilities in all cases.